ESAM 438
Interdisciplinary Nonlinear Dynamics

Fall 2000

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Problem Set 1

For Discussion Section October 4

1. **Some Short Nice Problems from Strogatz**

   Do 2.2.8, 2.2.10.

2. **Population Growth with Eggs**

   Consider the dynamics of a population of animales that lay eggs and investigate the following simple model. The animals propagate by laying eggs, which need a time τ to hatch. The rate at which they lay eggs is adversely affected if the population N is too dense, i.e. assume the rate decreases linearly in N. Similary the death rate increases linearly with N. Thus, you get

   $$\frac{dN}{dt} = -aN(t) - bN(t)^2 + \alpha N(t-\tau) - \beta N(t-\tau)^2. \tag{1}$$

   (a) Nondimensionalize the evolution equation by using the death rates as characteristic scales (in order to leave the delay as a control parameter).

   (b) Use the matlab program available on the class web site to solve first the simpler problem in which the delay vanishes, $\tau = 0$. Compare the numerical to the analytical solution by calculating

   $$E = \frac{\int_0^{t_{max}} (n_{ex}(t) - n_{num}(t))^2 \, dt}{\int_0^{t_{max}} (n_{ex}(t))^2 \, dt}, \tag{2}$$

   where n is the dimensionless density and t_{max} is chosen appropriately so that n has just about reached its stationary value. Measure the error E as a function of the time step, decreasing the time step Δt by factors of 2. Plot $E(\Delta t)$ double-logarithmically and confirm that for small Δt it is linear in Δt. Choose the time step small enough to get the error below 1%.

   (c) Modify the matlab program to solve the evolution equation for $\tau \neq 0$. The initial conditions require the knowledge of $N(t)$ for $-\tau < t < 0$. Use $N(t) = 0$ for that range of t (no eggs laid yet).

   (d) Study the dynamics of the equation numerically for $\tau = 3$ and $\beta = 3.7$ over a range of α between 10 and 20 (all in dimensionless quantities). Does the behavior change qualitatively when changing α? How does it differ from that of the simple logistic equation without delay? How is this behavior possible although the equation is only first-order in time?

   (e) Establish that the behavior you observe is not a numerical artifact by performing a convergence test for one representative case. Since you do not know the exact solution plot the difference $N(t_0, \Delta t) - N(t_0, \Delta t/2)$ for some suitably chosen fixed t_0 double-logaritmically[1] as a function of Δt and establish that this difference shows the correct scaling in Δt.

[1] Use the matlab command `loglog` instead of `plot`.

3. **Uniqueness of Solutions**

Consider

$$\frac{dx}{dt} = f(x). \tag{3}$$

(a) Show that for functions $f(x)$ that satisfy the Lipschitz condition in an interval $[x_0 - \Delta x, x_0 + \Delta x]$ around a stable fixed point x_0 the time to reach that fixed point is infinite. Discuss this result in view of the theorem stated in class about the uniqueness of solutions.

(b) In class it was stated that if $f'(x)$ is continuous the solution to (3) is unique. Discuss in this context the equation

$$\frac{dx}{dt} = |x|, \tag{4}$$

for which clearly $f'(x)$ is not continuous. Is the solution to this equation unique for all initial conditions? Why not? Why?