Interdisciplinary Nonlinear Dynamics (438)

Fall 2001

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Problem Set 1

For Discussion Section October 5

Note: the discussion section will be on Fridays 3:30-4:30 in M152. This week Sandeep will give an introduction to Matlab.

1. **Some Short Nice Problems from Strogatz**

   Do 2.2.9, 2.2.10.

2. **Population Growth with Eggs**

   Consider the dynamics of a population of animales that lay eggs and investigate the following simple model. The animals propagate by laying eggs, which need a time τ to hatch. The rate at which they lay eggs is adversely affected if the population N is too dense, i.e. assume the rate decreases linearly in N. Similary the death rate increases linearly with N. Thus, you get

$$\frac{dN}{dt} = -aN(t) - bN(t)^2 + \alpha N(t-\tau) - \beta N(t-\tau)^2. \tag{1}$$

   (a) Nondimensionalize the evolution equation by using the magnitude of the death rates $|a|$ and $|b|$ as characteristic scales (in order to leave the delay as a control parameter). In this process α, β, and τ will be rescaled to $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\tau}$. In the rest of the problem we will omit the $\tilde{}$ over the symbols again and write α, etc.

   (b) Use the matlab program available on the class web site to solve first the usual logistic equation, in which the delay vanishes, $\tau = 0$, and $a = -1$ and $b = 1$. Compare the numerical to the analytical solution by calculating

$$E^2 = \frac{\int_0^{t_{max}} (n_{ex}(t) - n_{num}(t))^2 \, dt}{\int_0^{t_{max}} (n_{ex}(t))^2 \, dt}, \tag{2}$$

   where n is the dimensionless density and t_{max} is chosen appropriately so that n has just about reached its stationary value. Measure the error E as a function of the time step, decreasing the time step Δt by factors of 2. Plot $E(\Delta t)$ double-logarithmically[1] and confirm that for small Δt it is linear in Δt. Choose the time step small enough to get the error below 1%.

   (c) To get a feeling for the *stability* of the simple forward Euler method used in the matlab program investigate the behavior of the numerical solution as you increase Δt in factors of 2 up to values of $\Delta t = 3$. What change in the behavior do you observe?

   (d) Modify the matlab program to solve the (dimensionless) evolution equation for $\tau \neq 0$. The initial conditions require the knowledge of $n(t)$ for $-\tau < t < 0$. Use $n(t) = 0$ for that range of t (no eggs laid yet).

   (e) Study the dynamics of the equation numerically for $\tau = 3$ and $\beta = 3.7$ over a range of α between 10 and 20 (all in dimensionless quantities). Does the behavior change qualitatively when changing α? How does it differ from that of the simple logistic equation without delay? How is this behavior possible although the equation is only first-order in time?

[1] Use the matlab command `loglog` instead of `plot`.

(f) Establish that the behavior you observe is not a numerical artifact by performing a convergence test for one representative case. Since you do not know the exact solution plot the difference $n(t_0, \Delta t) - n(t_0, \Delta t/2)$ for some suitably chosen fixed t_0 double-logaritmically as a function of Δt and establish that this difference shows the correct scaling in Δt.

3. **Uniqueness of Solutions**

Consider

$$\frac{du}{dt} = f(u). \tag{3}$$

(a) Show that for functions $f(u)$ that satisfy the Lipschitz condition in an interval $[u_0 - \Delta u, u_0 + \Delta u]$ around a stable fixed point u_0 the time to reach that fixed point is infinite. Discuss this result in view of the theorem stated in class about the uniqueness of solutions.

(b) In class it was stated that if $f'(u)$ is continuous the solution to (3) is unique. Discuss in this context the equation

$$\frac{du}{dt} = |u|, \tag{4}$$

for which clearly $f'(u)$ is not continuous. Is the solution to this equation unique for all initial conditions? Why not? Why?

4. **Adams-Bashforth and Predictor-Corrector Methods**

Consider the differential equation

$$\frac{du}{dt} = f(u). \tag{5}$$

The solution $u_{j+1} \equiv u(t + \Delta t)$ can be written exactly as

$$u_{j+1} = u_j + \int_{t_j}^{t_{j+1}} f(u(t))\, dt, \tag{6}$$

where $t_j = j\, \Delta t$.

(a) Approximate $f(u(t))$ by a polynomial in t, i.e. expand $f(u(t))$ in a Taylor expansion about t_j. Approximate the derivative $\frac{df}{dt}$ by a finite-difference expression involving only f at times $t \leq t_j$. Considering the term quadratic in Δt as an error term and inserting the expansion into the integral in (6) yields the *Adams-Bashforth* method. What order in Δt is the error of this time-stepping method?

(b) Use the trapezoidal method to approximate the integral in (6). This would require the knowledge of u_{j+1}, which is still unknown. Determine the approximation $\bar{u}$ for the u_{j+1} in the integral by a forward Euler step (this step is called the predictor step) and use it to determine then u_{j+1} (this step is called the corrector step). What order in Δt is the error for this *predictor-corrector* scheme?