Interdisciplinary Nonlinear Dynamics (438)
Fall 2001
Hermann Riecke

Problem Set 4

For Discussion Section October 26

1. **Dynamics Near Fixed Points**

   (a) Calculate the eigenvectors and eigenvalues of

   $$\frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x}(t) \tag{1}$$

   and use them to give the general solution to (1).

   (b) Calculate the general solution $\mathbf{x}(t)$ (for general initial condition $\mathbf{x}_0$) of

   $$\frac{d\mathbf{x}(t)}{dt} = \begin{pmatrix} -\lambda-\epsilon & 1 \\ 0 & -\lambda \end{pmatrix} \mathbf{x}(t) \tag{2}$$

   with $\lambda > 0$ and $\epsilon > 0$. Determine from it a general orbit in the phase plane. Sketch the phase portrait of this system.
   Sketch now the phase portrait for a degenerate node, i.e. consider the limit $\epsilon \to 0$ of (2). Again obtain first the general solution and a general orbit.

2. **Phase Portraits**

   Do Strogatz 6.1.3, 6.1.6, 6.1.7, 6.1.12.

3. **Different Types of Stability**

   (a) Give an example of a two-dimensional system that is Lyapunov stable but not asymptotically nor linearly stable.

   (b) Give an example of a system that is asymptotically stable but not linearly stable.

   (c) Give an example of a system that is asymptotically unstable but not linearly unstable.

4. **Population Growth with Eggs III**

   Consider

   $$\frac{dN}{dt} = -N(t) - N(t)^2 + \alpha N(t-\tau) - \beta N(t-\tau)^2. \tag{3}$$

   To get a better overview of the possible behavior of that equation investigate it analytically.

   (a) Determine all fixed points N_0 of the equation and linearize around a non-trivial fixed point $N_0 \neq 0$,

   $$N = N_0 + N_1\, e^{\lambda t}, \tag{4}$$

   to obtain an equation for the growth rate λ.

   (b) Can this fixed point undergo an instability with respect to a monotonic perturbation, i.e. with a real λ going through 0? If so, determine the onset of the instability numerically[1].

[1] You can use `fzero` in matlab.

(c) Can this fixed point undergo an instability with respect to an oscillatory perturbation, i.e. with the real part of the complex λ going through 0? If so, determine the onset of the instability numerically. Note that there may occur more than one instability. How can this system support oscillations although the differential equation is only first-order in time?

(d) Do your analytical results match your findings in the numerical study of (3) performed in the previous homework?

5. **Poincaré-Bendixson Theorem**

Consider the nonlinear system

$$\frac{dx}{dt} = y + x^3 - x\,(x^2 + y^2)^2, \tag{5}$$

$$\frac{dy}{dt} = -x + y^3 - y\,(x^2 + y^2)^2. \tag{6}$$

(a) Perform a linear stability analysis of the fixed point $(0, 0)$ and use the Poincaré-Bendixson Theorem to show that (5,6) has a limit cycle.

(b) Solve (5,6) numerically[2] to check your result from a). Plot representative orbits to demonstrate your result. Is the orbit stable or unstable?

[2] You may want to go from the Euler scheme used in the previous problems to a 4th-order Runge-Kutta scheme.