Interdisciplinary Nonlinear Dynamics (438)

Fall 2001

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Problem Set 7

For Discussion Section November 30

1. **An Interesting Dynamical System II**

   Consider again the system

$$\frac{dx}{dt} = y, \tag{1}$$
$$\frac{dy}{dt} = \alpha x + \beta y + a x^3 - x^2 y. \tag{2}$$

   from the previous homework set.

   (a) Consider now the case $a = +1$.
   Choose $\alpha = -1$ and increase β across the instability identified above. What happens in the numerical simulation?
   Measure the amplitude $\mathcal{A}$ and the frequency ω of the oscillations as a function of β, increasing β up to $\beta = 0.2$. Is there a range in β over which the dependence of $\mathcal{A}$ and ω on β agrees with your expectation for such a bifurcation?
   What happens for $\beta > 0.2$?

   (b) Go back to the case $a = -1$.
   Study numericaly how the phase portrait changes for $\alpha = 0.1$ when β is varied from $\beta = -1$ to $\beta = 0.2$. What happens to the center manifold?

2. **Pitch-Fork Bifurcation by Multiple Times**

   Consider again (1,2). Use now multi-timing to analyze the steady bifurcation, which you analyzed in Problem 1.b.ii of the last problem set using center-manifold reduction. Do you get the same bifurcation equation?

3. **Swift-Hohenberg Equation without Up-Down Symmetry**

   Consider the modified Swift-Hohenberg equation

$$\partial_t \psi = \mu\psi - (\partial_x^2 + 1)^2\psi + \alpha\psi^2 - \psi^3, \qquad -\infty < x < \infty, \tag{3}$$

   which does not possess the 'up-down' symmetry $\psi \to -\psi$. Derive the Ginzburg-Landau equation for (3) by expanding around the periodic solution with critical wavenumber $q_c = 1$ and allowing the amplitude to be a function of slow time *and* space. Show that one obtains a pitch-fork bifurcation despite the broken symmetry. Check that the spatial translation symmetry of (3), i.e. invariance under $x \to \Delta x$ for any Δx, corresponds (implies) the symmetry of the Ginzburg-Landau equation under phase shifts $A \to Ae^{i\Delta\phi}$. This illustrates that the origin of the pitch-fork bifurcation is the translation symmetry of the system.

4. **Numerical Solution of Ginzburg-Landau Equation**

   (a) Modify the code provided on the class web site, which solves the SH-equation, to solve the Ginzburg-Landau equation

$$\partial_t A = (1 + ic_1)\partial_x^2 A + \mu A - (1 - ic_3)|A|^2 A, \tag{4}$$

where A is the complex amplitude of the pattern. The code uses backward Euler[1] for the spatial derivatives, whereas the terms without derivatives are updated with a forward Euler step. The backward Euler method provides unconditional stability with respect to the derivative terms. Thus, you want to modify the code to solve

$$A_j^{n+1} = A_j^n + (1 + ic_1)\frac{\Delta t}{\Delta x^2}\left(A_{j+1}^{n+1} - 2A_j^{n+1} + A_{j-1}^{n+1}\right) + \Delta t\left(\mu A_j^n - (1 + ic_3)|A_j^n|^2 A_j^n\right). \quad (5)$$

Note that matlab actually deals with complex variables in a straightforward manner. Plot the real part of A, the imaginary part of A, and its absolute value.

(b) Use your code to study numerically the Eckhaus stability limits obtained in class from the phase equation. Do that for the case $c_1 = 0 = c_3$, i.e. for steady patterns. Start with a *periodic* pattern in a sufficiently large system (at least 10 to 15 wavelengths) in the stable regime and let it converge to a steady solution. Then decrease μ somewat below the expected stability limit and perturb this non-zero solution by adding some noise. How does this perturbed solution evolve in time and what is the final outcome of the simulation?

(c) Just for the fun of it run the code now for $c_1 = 0$ and $c_3 = 1.4$ with some random initial condition. In this regime the complex Ginzburg-Landau equation shows interesting behavior (cf. M. van Hecke, Phys. Rev. Lett. 80 (1998) 1896, go to `prl.aps.org` to look at the paper on the web. More interesting regimes can be found in H. Chaté, Nonlinearity 7 (1994) 185 (`http://www.iop.org/Journals/no`)).

[1] Setting $\alpha = 0.5$ in the code turns it into a Crank-Nicholson scheme.