Contents

1 Introduction

Dynamics arise in many systems

- Mechanics: vibrations, coupled structural elements

  driven $\Rightarrow$ complex behavior even in simple driven pendulum

  planetary motion: n-body problem

  planetary system stable?

- Fluids: Rayleigh-Bénard $\rightarrow$ WWW
  transitions between different states
  transition to turbulence, channel flow

- Chemical systems: BZ reaction $\rightarrow$ WWW: spiral waves
  flames $\rightarrow$ WWW

- Economics: pig cycle, delay between price and investment

  job market: delay by education
  increase demand: more freshmen
  4 years later too many applicants
  $\rightarrow$ fewer freshmen

- Dynamics in the heart:

  contraction = excitation of muscle (wave at ball game, excitable)
  propagates like a wave
  defects in waves: spirals, diseases
  fibrillations

  WWW: rabbit heart picture

Nonlinear Systems:

- changes in qualitative behavior

  non-smooth dependence on parameters

  picture: TVF: torque & flow pictures

- multiplicity of solutions: hysteresis

  rolls vs. SDC convection

- chaotic dynamics

  (many frequencies, coexisting (unstable) periodic solutions)

Simple illustration: linear vs. nonlinear

Consider $\quad f(x, \mu) = 0$

..(essentially) always	..# of solutions can change with μ.
1 unique solution	solutions disappear and appear
quantitative but	**quantitative** and **qualitative** changes
no qualitative change	

Example: TVF, torque & flow pictures, multiplicity, phase diagram (WWW)

Mathematical Description of Dynamics: Differential Eqn. & Maps

ordinary differential equations (ODE) : coupled oscillators
partial differential equations (PDE) : heat eqn., reaction diffusion eqn. Navier-Stokes
Maps : stroboscopic description

Linear:

$$\ddot{\mathbf{u}} = \mathbf{A}\mathbf{u}$$

superposition of solutions to get **general solution** for **all** initial conditions

Notation: $\dot{u} \equiv \frac{du}{dt}$, temporal derivative

Example: 2 masses with springs, u = transverse elongation
parallel and anti-parallel mode (lower and higher frequency)
any motion = parallel + anti-parallel

PDE: Fourier expansion

$$\begin{aligned} \ddot{u} &= u'' \quad 0 < x < L \quad \text{string} \\ u &= \sum_{n=-\infty}^{\infty} u_n e^{i\frac{2\pi}{L}nx} \end{aligned}$$

Notation: $u' = \frac{du}{dx}$, spatial derivative

eigenmodes: each u_n satisfies: $\ddot{u}_n = -(\frac{2\pi n}{L})^2 u_n \rightarrow u_n = \psi_n(0)e^{i\frac{2rn}{L}t}$

different modes do **not** interact

Nonlinear: no superposition, different modes do interact!

$$\ddot{u} = u'' + \underbrace{u^2}_{\sum u_n u_m e^{i\frac{2\pi}{L}(n+m)x}}$$

u^2 generates new wave number: couples n & m to $n+m$

Any interaction between different objects (A and B) implies nonlinearity:
evolution of A depends on state of A and that of B

$\rightarrow$ Cannot build general solution from a set of basic solutions

$\rightarrow$ in general: cannot find exact solutions: **HARD**.

Numerical Solution:

- confirms the model/basic equations: of great interest if model has not been established, e.g., chemical oscillations, heart muscle
- gives quantitative details for **specific** values of system parameters:
  these details may not be accessible in experiments: 3d fluid flow, turbulent

  We'll do some numerical things

Overview and Insight: Qualitative Analysis

- change in behavior as system parameters are changed
  transitions between **qualitatively** different states
- analytical techniques for transitions
  approximations near transition points
- visualization: geometry of dynamics
  phase space
- overview over all possible behaviors

Example: mass-spring-system

$$\dot{x} = v$$
$$\dot{v} = -\frac{K}{m}x$$

Figure

for **any** initial condition simple periodic motion:
complete overview of all possible solutions
(here linear $\rightarrow$ not much going on.)

Conservative vs. Dissipative

with friction

Figure

- range of initial conditions leads to same state: **attractors**
- **transitions**: **qualitative** change in attractors

We will mostly focus on dissipative systems.

2 1-d Flow

2.1 Flow on the Line

$$\dot{x} = f(x)$$

can be solved exactly for any $f(x)$

$$\int_{x_0}^{x} \frac{dx}{f(x)} = t - t_0$$

Example: $\dot{x} = \sin x$

$$t = \int \frac{dx}{\sin x} = \int \csc x' dx'$$
$$t = -\ln|\csc x + \cot x| + C$$

Now what? What have we learned?

Even if we could solve for x, would we have an overview of behavior of system for arbitrary initial conditions?

Geometrical picture: phase space

`Figure`

$\dot{x} = f(x)$ defines flow in phase space or a vector field

For 1d: plot $f(x)$ in addition

`Figure`

Conclude: any i.c. ends up in one of the **fixed points** at $x = (2n+1)\pi$: **stagnation point** of flow

stability:

- flow into $(2n+1)\pi$: stable
- flow out of $2n\pi$: unstable

Of course: for **numbers** we need the detailed solution

Example: Population growth with Limited Resources

$$\dot{N} = \underbrace{g(N)}_{\text{combined birth and death rate}} \overbrace{N}^{N = \# \text{ of animals}}$$

limited food/space: $g = \alpha - \underbrace{\beta N}_{\text{more deaths less births}}$

Logistic growth model

$$\dot{N} = \alpha N - \beta N^2$$

Make dimensionless: $[\alpha] = \frac{1}{S}$ $[\beta] = \frac{1}{S}\frac{1}{\#}$

$$\tau = \alpha t \qquad n = \frac{\beta}{\alpha} N$$

$\frac{1}{\alpha}$ characteristic growth time
$\frac{\alpha}{\beta}$ characteristic population site

Question: If population goes to some equilibrium, what size would you expect?

$\frac{\alpha}{\beta}$ is the only characteristic size after initial condition is forgotten $\Rightarrow$ expect N $\to \frac{\alpha}{\beta}$

$$\partial_\tau n = n - n^2$$

Could solve by partial fraction

Phase space:

Figure

fixed points: $n = 0$, $n = 1$

Flow indicates: $n = 0$ unstable, $n = 1$ stable

indeed: all i.c. go to $N = \frac{\alpha}{\beta}$.

— stuff skipped —

3 Two-dimensional Systems

new aspects:

- 'true' oscillations without periodic "boundary" conditions
- reduction of dynamics to lower dimension

3.1 Classification of Linear Systems

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} \qquad \underline{x}(0) = \underline{x}_0$$

formal solution

$$\underline{x}(t) = e^{\underline{\underline{A}}\,t}\underline{x}_0$$

with

$$e^{\underline{\underline{A}}\,t} = 1 + \underline{\underline{A}} + \frac{1}{2}\underline{\underline{A}}^2 t^2 + \ldots$$

Simplify $\underline{\underline{A}}$ by similarity transformation:

in general can find $\underline{\underline{S}}$ such that

$$\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

or

$$\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{Jordan normal form}$$

Notes:

- $\lambda_{1,2}$ are the eigenvalues of $\underline{\underline{A}}$:

$$\begin{aligned} \underline{\underline{A}}\,\underline{v}_{1,2} &= \lambda_{1,2}\underline{v}_{1,2} \\ \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \lambda_1 \begin{matrix} 1 \\ 0 \end{matrix} \\ \Rightarrow \underline{\underline{A}}\,\underbrace{(\underline{\underline{S}}\,(\substack{1\\0}))}_{\underline{v}_1} &= \lambda_1(\underline{\underline{S}}\,(\substack{1\\0})) \end{aligned}$$

- all eigenvalues different

$$\Rightarrow \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} \quad \text{diagonal}$$

- dynamics in eigendirections simple

$$\begin{aligned} e^{\underline{\underline{A}}\,t}\underline{v}_i &= \{1 + \underline{\underline{A}}\,t + \frac{1}{2}(\underline{\underline{A}}\,t)^2 + \cdots\}\underline{v}_i = \\ &= \{1 + \lambda_i t + \frac{1}{2}\lambda_i^2 t^2 + \cdots\}\underline{v}_i = \\ &= e^{\lambda_i t} \end{aligned}$$

  along eigendirections simple exponential time dependence

- general solution

$$\underline{x}(t) = e^{\lambda_1 t}\underline{v}_1 A_1 + e^{\lambda_2 t}\underline{v}_2 A_2$$

  with $\underline{x}_0 = A_1\underline{v}_1 + A_2\underline{v}_2$

- eigenvalues can be complex

$$\begin{aligned} \lambda_{1,2} &= \sigma \pm i\omega \\ \underline{x}(t) &= e^{\sigma t}(A_1 e^{i\omega t}\underline{v}_1 + A_2 e^{-\sigma t}\underline{v}_2) \end{aligned}$$

- degenerate eigenvalues → modifications, see later
- calculation of eigenvalues in 2d:

$$\det \underline{\underline{A}} = \det(\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}) = \lambda_1\lambda_2 \qquad tr\underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}} = \lambda_1 + \lambda_2$$

  quadratic formula

$$\lambda_{1,2} = \frac{+tr\underline{\underline{A}} \pm \sqrt{(tr\underline{\underline{A}})^2 - 4\det\underline{\underline{A}}}}{2}$$

Orbits in phase space (plane):

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{c} x = e^{\lambda_1 t}x_0 \\ y = e^{\lambda_2 t}y_0 \end{array}$$

$$\Rightarrow e^t = (\frac{x}{x_0})^{1/\lambda_1}$$

$$y(t) = \left(\left(\frac{x}{x_0}\right)^{1/\lambda_1}\right)^{\lambda^2} y_0 = y_0\left(\frac{x}{x_0}\right)^{\frac{\lambda_2}{\lambda_0}}$$

$$y(t) = Cx(t)^{\frac{\lambda_2}{\lambda_1}}$$

`Figure`

`Figure`

Definition: Stable/unstable manifold $\{\underline{x}\,|\,\underline{x}(0) = \underline{x} \Rightarrow \underline{x}(t) \to \underline{x}_0$ for $t \to \pm\infty\}$

Note: · in general eigenvectors need not be orthogonal

ii)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

eigenvalues

$$\det\begin{vmatrix} +3-\lambda & -2 \\ -1 & 2-\lambda \end{vmatrix} = 6 - 5\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-16}}{2} = \begin{cases} +4 \\ +1 \end{cases}$$

eigenvectors:

$$\lambda_1 = 4 : \qquad 3x - 2y = 4x \;\;\Rightarrow\;\; y = -\frac{1}{2}x \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$$\lambda_2 = 1 : \qquad 3x - 2y = x \;\;\Rightarrow\;\; y = x \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

`Figure`

Possible Phase Portraits

`Figure`

Change in stability: Re $(\lambda_i) = 0$

i) $tr\underline{\underline{A}} = 0$ and $\det \underline{\underline{A}} > 0 \Rightarrow$ complex pair $\lambda = \pm i\omega$

ii) $tr\underline{\underline{A}} < 0$ and $\det \underline{\underline{A}} > 0 \Rightarrow \lambda_1 = 0 \;\; \lambda_2 = 0$

Change in character:

$$\text{real} \mapsto \text{complex}$$

$$tr\underline{\underline{A}}^{\,2} = 4 \det \underline{\underline{A}}$$

`Figure`

Notes:

- degenerate node $\Rightarrow$ border between nodes and spirals, does not quite oscillate
- non-isolated fixed points: steady bifurcation one or more fixed points are created/annihilated (details depend on nonlinearities)

3.2 Stability

So far we had linear stability

- with respect to infinitesimal perturbations
- determined by linearization

Example: damped-driven pendulum

$$m\ell^2\ddot{\theta} + \beta t\dot{h}eta = -mg\ell\sin\theta + \Gamma$$

write as first-order system:

$$\begin{aligned}\dot{x} &= y \equiv F_x(x_1 y)\\ \dot{y} &= -\frac{\beta}{m\ell^2}y - \frac{mg\ell}{m\ell^2}\sin x + \Gamma \equiv F_y(x_1 y)\end{aligned}$$

Fixed points:

$$y_0 = 0 \qquad \& \qquad mg\ell\sin x_0 = \Gamma$$

Expand:

$$\begin{aligned}x &= x_0 + \epsilon x_1(t) \qquad \epsilon \ll 1\\ y &= y_0 + \epsilon y_1(t)\end{aligned}$$

$$\begin{aligned}\epsilon\dot{x}_1 &= F_x(x_0 + \epsilon x_(1,1) y_0 + \in y_1) =\\ &= \underbrace{F_x(x_0, y_1)}_{0} + \epsilon x_1 \partial_x F_x|_{(x_0,y_0)} + \epsilon y_1 \partial_y F_x F_x|_{(x_0,y_0)} + \sigma(\epsilon^2) \begin{pmatrix}\dot{x}_1\\ \dot{y}_1\end{pmatrix} \begin{pmatrix}\partial_x F_x & \partial_y F_x\\ \partial_x F_x & \partial_y F_x\end{pmatrix} \begin{pmatrix}x_1\\ y_1\end{pmatrix}\end{aligned}$$

$\Rightarrow$ linear stability determined by eigenvalues of Jacobian here:

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1\\ \frac{-g}{\ell}\cos x_0 & -\frac{\beta}{m\ell^2}\end{pmatrix}$$

eigenvalues: $(-\lambda)(-\lambda\beta_{m\ell^2}) + \frac{g}{\ell}\cos x_0 = 0$

$$\begin{aligned} \lambda^2 &+ \lambda\frac{\beta}{m\ell^2} + \frac{g}{\ell}\cos x_0 = 0 \\ \lambda_{1,2} &= \frac{\beta}{m\ell^2} \pm \frac{1}{2}\sqrt{(\frac{\beta}{m\ell^2})^2 - 4\frac{g}{\ell}\cos x_0 = 0} \end{aligned}$$

Figure

All orbits that start sufficiently close to fixed point remain close to it for all times

technically: $x_{Fp} \in U \subseteq V : x(0) \in U \Rightarrow x(t) \in V$ for all t

Asymptotic stability:

All orbits that start sufficiently close to fixed point converge to it as $t \to \infty$.

Figure

Notes:

- asymptotically stable $\Rightarrow$ fixed point is attracting, it is an attractor.
- linear stability $\Rightarrow$ asymptotic stability
  $\Rightarrow$ Lyapunov stability
- linear instability $\Rightarrow$ instability
- But: asymptotic or Lyapunov stability do not imply linear stability

Examples:

- center is Lyapunov stable, but linearly neither stable nor unstable (marginally stable
- stability can be determined purely by nonlinear terms

$$\dot{x} = \alpha x^3$$

$x = 0$ linearly marginally stable

$$x(t) = \pm\sqrt{\frac{x_0^2}{1 - 2x_0^2\alpha t}}$$

$\Rightarrow$ $\alpha > 0$ (nonlinearly) unstable
$\alpha < 0$ (nonlinearly) asymptotically stable

3.3 General Properties of Phase Plane

3.3.1 Phase Portrait

A phase portrait captures all relevant features of the phase plane:

- fixed points
  stable/unstable manifolds
- periodic orbits
- separatrices and other additional orbits that visualize the flow

Example

Figure

Notes:

- For $\underline{\dot{x}} = \underline{f}(\underline{x})$ solution unique of all $\frac{\partial f_i}{\partial x_j}$ are continuous
  $\Rightarrow$ orbits do not intersect.

Figurenonunique

- Separatix separates basins of attraction of different attractors

more examples for stating phase portraits in homework.

3.3.2 Hartman-Grobman theorem

Linear systems: complete understanding

How much of that can be transferred to nonlinear systems?

Definition: A fixed point x_0 of $\underline{\dot{x}} = \underline{f}b(\underline{x})$ is called hyperbolic if all eigenvalues of $\frac{\partial f_i}{\partial x_j}$ have non-zero real parts.

Thus: in all directions a hyperbolic fixed point is either linearly attractive or repulsive. No marginal direction.

Hartman-Grobman Theorem:

If x_0 is a hyperbolic fixed point of $\underline{\dot{x}} = \underline{f}(\underline{x})$ then there exists a continuous invertible function $\underline{h}(\underline{x})$ that is defined on some neighborhood of $\underline{x}_0$ and maps all orbits of the nonlinear flow into those of the linear flow. The map can be chosen so that the parameterization of orbits by time is preserved.

Figure

Thus:

- For hyperbolic fixed points $\underline{x}_0$ the linearization of the flow gives us the topology of the nonlinear flow in a neighborhood of $\underline{x}_0$.

Note:

- If fixed point is not hyperbolic, linearization does not give sufficient information:

$$\dot{x} = \alpha x^3$$

Figure

- At any bifurcation the fixed point is not hyperbolic.

  $\dot{x} = \mu x + \alpha x^3$, at $\mu = 0$ linear systems equal.

Note:

- the flow in the vicinity of a hyperbolic fixed point is structurally stable.. (not the case without hyperbolicity, e.g., centers, fixed points undergoing bifurcations.)

3.3.3 Ruling Out Oscillations

For what kind of systems can one rule out persistent dynamics like periodic orbits?

i) Gradient Systems, Potential Systems

$$\dot{\underline{x}} = -\nabla V(\underline{x})$$

or

$$\dot{x}_i = \frac{\partial v}{\partial x_i}$$

with $V \geq V_0$ for all $\underline{x}$ (bounded from below)

$$\Rightarrow \frac{dV}{dr} = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i = -\sum_i (\frac{\partial V}{\partial x_i})^2 \leq 0$$

thus eventually V reaches a (local) minimum

then:

$$\frac{dV}{dt} = 0 \Leftrightarrow \frac{\partial V}{\partial x_i} = 0 \cos \dot{x}_i - 0$$

reaches some fixed point.

Example: Mechanical overdamped particle in potential

`Figure`

<u>ii) Lyapunov Functional</u>

need not $\dot{\underline{x}} = -\nabla V$

sufficient for ruling out periodic orbits:

Assume $V(\underline{x}) > V_o$ for all $\underline{x} \neq \underline{x}_0$ fixed point

- if $\frac{dV}{dt} \leq 0$ for all $\underline{x} \neq \underline{x}_0$ in neighborhood $\mathcal{U}$
  then $\underline{x}_0$ Lyapunov stable
- if $\frac{dV}{dt} < 0$ for all $\underline{x} \neq \underline{x}_0$ in $\mathcal{U}$
  then $\underline{x}_0$ asymptotically stable

Note: Such a V is called a Lyapunov functional.

Example:

a) damped particle in bounded potential

$$\ddot{x} + \beta \dot{x} = -\frac{d\mathcal{U}}{dx}$$

i.e.

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\beta v - \frac{d\mathcal{U}}{dx} \end{aligned}$$

Try $V = \frac{1}{2}\dot{x}^2 + \mathcal{U} = \frac{1}{2}v^2\mathcal{U}$

$$\frac{dV}{dt} = v\dot{v} + \frac{d\mathcal{U}}{dx}\dot{x} = v(-\beta v - \frac{d\mathcal{U}}{dx}) + \frac{d\mathcal{U}}{dx}v = -\beta v^2 < 0$$

$\Rightarrow$ fixed points asymptotically stable, no periodic orbits.

b)

$$\begin{aligned}\dot{x} &= -x + 4y \\ \dot{y} &= -x - y^3\end{aligned}$$

Simplest attempt: try quadratic function that is bounded from below:

$$V = x^2 + ay^2 \quad \text{with } a > 0.$$

$$\begin{aligned}\frac{dV}{dt} &= 2x(-x + 4y) + 2\alpha y(-x - y^3) \\ &= \underbrace{-2x^2}_{\leq 0} \; \underbrace{xy(8 - 2a)}_{\text{undetermined}} \; - \underbrace{2ay^4}_{\leq 0}\end{aligned}$$

$\Rightarrow$ choose $a = 4 \Rightarrow \frac{dV}{dt} < 0$ for $x \neq 0 \neq y$

$\Rightarrow$ $(0, 0)$ asymptotically stable

and no periodic orbits

Note: potentials rule out persistent dynamics in arbitrary dimensions.

3.3.4 Poincaré-Bendixson Theorem

Can we guarantee a periodic orbit without explicitly calculating it?

Poincaré-Bendixson Theorem

Assume

- R is a closed bounded subset of the plane

- $\dot{\underline{x}} = \underline{f}(\underline{x})$ with $\underline{f}(\underline{x})$ continuously differentiable on an open set containing R

then

> any orbit that remains in R for all t either converges to a fixed point or to a periodic orbit.

Pictures:

R can be considered as a trapping region

if orbit continues inside the other part it will have to go to periodic orbit (of fixed point)

to avoid continuing on one side of the other part

more and more wiggles:

to continue to $t \to \infty$ need infinitely many wiggles (otherwise fixed point)

velocity transverse to ?

`Figure`

domain R bounded $\Rightarrow$ wiggles become dense, pile up at a $x = x_{\max}$ which is inside R $x_{\max} \in R$

$\underline{f}(\underline{x})$ continuously differentiable

$\Rightarrow$ amplitude has to go to 0 at $x_{\max}$

$$\begin{aligned} f &\sim (x - x_{\max})^{\alpha} \cos\left(\frac{1}{x - x_{\max}}\right) \\ f' &\sim \alpha(x - x_{\max})^{\alpha-1} \cos\left(\frac{1}{x - x_{\max}}\right) + (x - x_{\max})^{\alpha} \sin\left(\frac{1}{x - x_{\max}}\right) \cdot \left(\frac{1}{x - x_{\max}}\right)^2 \end{aligned}$$

f' continuous at $x = x_{\max}$
$\Rightarrow$ need $f' \to 0 \Rightarrow \alpha - 2 > 0$

$\Rightarrow$ has to converge to fixed point since $f \to 0$ inside domain R.

In other words: any more complicated dynamics would require flow to change orientation on small scales: not possible for sufficiently nice vector fields $\underline{f}(\underline{x})$.

Example: Glycolysis Oscillations

yeast cells break down sugar by glycolysis simple model

ADP adenoside diphosphate	x
FGP fructose-6-phosphate	y

$$
\begin{aligned}
\dot{x} &= -x + ay + x^2 y = f(x,y) \\
\dot{y} &= b - ay - x^2 y = g(x,y)
\end{aligned}
$$

<u>Phase portrait:</u>

study <u>nullclines</u>: $\dot{x} = 0$ or $\dot{y} = 0$

$$
\begin{aligned}
f = 0 \qquad & \Rightarrow y = \frac{x}{a + x^2} \\
g = 0 \qquad & \Rightarrow y = \frac{b}{a + y^2}
\end{aligned}
$$

$\Rightarrow$ fixed point at

$$
\frac{x}{a + x^2} = \tfrac{b}{a+x^2}
$$

$$
\Rightarrow \begin{array}{rcl} x &=& b \\ y &=& \frac{b}{a+b^2} \end{array}
$$

exists for all $b > 0, a > 0$

`Figure`

null clines show: spiraling motion

- to fixed point?
- to periodic orbit? which?
- to infinity?

to use Poincaré-Bendixson:

i) need trapping region $\mathcal{R}$

ii) exclude fixed points from trapping region

`Figure`

Consider large x and y (check possibility of escape)

$$\left.\begin{array}{rcl} \dot{x} & \sim & x^2y \\ \dot{y} & \sim & -x^2y \end{array}\right\} \text{along orbit} \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \sim -|$$

compare $|\dot{x}|$ with $|\dot{y}|$ more precisely

$$\begin{array}{rcl} \dot{x} - (-\dot{y}) & = & -x + ay + x^2/y + b - ay - x^2/y \\ & = & b - x \end{array}$$

$\Rightarrow$ for $x > b$ $\qquad |\dot{x}| < |\dot{y}|$

for $y > \frac{b}{a} g < 0$ along $y = -x + C$ flow inward for $x > b$

`Figure`

ii) only a single fixed point $\left(b, \frac{b}{a+b^2}\right)$

Stability analysis shows

unstable for

$$1 - 2a - \sqrt{1-8a} < 2b^2 < 1 - 2a + \sqrt{1-8a}$$

$\Rightarrow$ limit cycle guaranteed for this range of b (if $a \leq \frac{1}{8}$)

formally: trapping region needs to exclude (small domain around fixed point)

outside this range expect conveyence to fixed point.

<u>Consequence of Poincaré-Bendixson:</u>

2s-flows have at most

- fixed points
- periodic orbits

as attractors.

No chaos in 2 dimensions.

Since: no crossing of orbits

need 3^{rd} dimension to get out of spiral.

3.4 Weakly Nonlinear Oscillators

Exact nonlinear solutions usually impossible to get.
Develop techniques to calculate explicit approximate periodic orbits and transients approaching periodic orbits.

3.4.1 Failure of Regular Perturbation Theory

Consider simple linear example to demonstrate problem

$$\ddot{x} + 2\epsilon\beta\dot{x} + (1+\epsilon\Omega)^2 x = 0$$

with some initial condition like $x(0) = 0$, $\dot{x}(0) = 1$.

Exact solution:

$$x \sim e^{\lambda t} \qquad \Rightarrow \qquad \lambda^2 + 2\epsilon\beta\lambda + (1+\epsilon\Omega)^2 = 0$$

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2\epsilon\beta \pm \sqrt{4\epsilon^2\beta^2 - 4(1+\epsilon\Omega)^2}}{2} \\
&= \pm i\sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2} - \epsilon\beta \\
\Rightarrow & \\
x_{exact} &= e^{-\epsilon\beta t}\left(Ae^{i\omega t} + A^* e^{-i\omega t}\right) \\
\text{with} & \\
\omega &= \sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2}
\end{aligned}$$

Attempt perturbation solution

$$\begin{aligned}
\frac{d^2}{dt^2}(x_0 + \epsilon x_1 + \ldots) &+ 2\epsilon\beta\frac{d}{dt}(x_0 + \epsilon x_1 + \ldots) \\
&+ (1+\epsilon\Omega)^2(x_0 + \epsilon x_1 + \ldots) = 0
\end{aligned}$$

Insert:
$0(\epsilon^0)$:

$$\frac{d^2}{dt^2}x_0 + x_0 = 0$$

$$x_0 = Ae^{it} + A^*e^{-it} = 2A_r \cos t - 2A_i \sin t$$

$0(\epsilon^1)$:

$$\frac{d^2}{dt^2}x_1 + 2\beta\frac{d}{dt}x_0 + 2\Omega x_0 + x_1 = 0$$

$$\frac{d^2}{dt^2}x_1 + x_1 \quad = \quad -2\underbrace{\left(-2i\beta Ae^{it} - 2\Omega Ae^{it}\right)}_{\sim\text{resonant forcing}} + c.c.$$

This is a second-order constant-coefficient *inhomogeneous* differential equation:
General solution: $x_1(t) = x_{homo}(t) + x_{particular}(t)$
with

$$\frac{d^2}{dt^2}x_{homo} + x_{homo} = 0 \qquad \Rightarrow x_{homo} = A_1e^{it} + c.c.$$

Guess ('ansatz') for particular solution (since inhomogeneity is simple exponential function):

$$x_{particular} = Be^{it} + c.c.$$

However:

$$\frac{d^2}{dt^2}Be^{it} + Be^{it} = 0 \qquad \Rightarrow \text{cannot balance inhomogeneity}$$

Could use method of *variation of constants* $x_{particular} = B(t)e^{it}$ and reduce the order of the equation and solve the resulting first-order equation by integration.

Here try ansatz:

$$x_{particular} = B\,t\,e^{it} + c.c.$$

Insert:

$$\begin{aligned}
\frac{d^2}{dt^2}x_{particular} + x_{particular} &= \\
B\left(0\,e^{it} + 2ie^{it} + (i)^2\,t\,e^{it} + t\,e^{it}\right) + c.c. &= -2\left(-2i\beta Ae^{it} - 2\Omega Ae^{it}\right) + c.c. \\
\Rightarrow B &= \frac{1}{2i}\left(-2i\beta - 2\Omega\right)A
\end{aligned}$$

Notes:

- resonant forcing leads to (linear) growth without bounds: secular terms
- solution breaks down for $t = 0(\frac{1}{\epsilon})$

Compare with exact solution

`Figure`

Approximation grows (instead of decay) and has wrong frequency.

But: approximation is expansion of exact solution in ϵ.

$$x_{exact} = \underbrace{e^{-\epsilon\beta t}}_{1-\epsilon\beta t+O(\epsilon^2)} \left(Ae^{i\omega t} + c.c.\right)$$

with

$$\omega = \underbrace{\sqrt{(1+\epsilon\Omega)^2 - \epsilon^2\beta^2}}_{1+\epsilon\Omega+O(\epsilon^2)}$$

$$x_{exact} = Ae^{it} + \epsilon\left(-\beta t + i\Omega t\right) + O(\epsilon^2) + c.c.$$

Thus:

- Straightforward perturbation expansion misses
  - slow growth/decay
  - small change in frequency
- secular terms suggest what true solution is doing.

3.4.2 Multiple Scales

Exact solution suggests that there are multiple time scales

$$t \qquad \text{and} \qquad T = \epsilon t \qquad \text{and} \qquad T_2 = \epsilon^2 t \ldots$$

$$x_{exact} = A\sin(t + \Omega T + \ldots)e^{-\beta T}$$

$\Rightarrow$ try $\qquad x = x_0(t, T_1, T_2 \ldots) + \epsilon x_1(t, T_1, T_2 \ldots) + \ldots$

Note:

- in this approach the two (or more times) are treated as essentially independent variables

$$\frac{d}{dt}x = \partial_{\hat{t}} x \frac{d\hat{t}}{dt} + \partial_T x \frac{dT}{dt} + \cdots = \left(\partial_{\hat{t}} + \epsilon\partial_T + O(\epsilon^2)\right) x$$

Redo same linear problem:

$$\begin{aligned}(\partial_{\hat{t}} + \epsilon\partial_T + \cdots)^2 (x_0 + \epsilon x_1 + \cdots) \quad &+ \quad 2\epsilon (\partial_t + \epsilon\partial_T + \cdots)(x_0 + \epsilon x_1 + \cdots) \\ &+ \quad (1 + \Delta\epsilon)(x_0 + \epsilon x_1 + \cdots) = 0\end{aligned}$$

$0(\epsilon^0)$:

$$\frac{d^2}{dt^2}x_0 + x_0 = 0$$

$$x_0 = Ae^{it} + A^* e^{-it} = 2A_r \cos t - 2A_i \sin t$$

Note:

- now A is not constant: $A = A(T, T_2, \ldots)$

$0(\epsilon^1)$:

$$2\partial_{\hat{t}}\partial_T x_0 + \partial_{\hat{t}}^2 x_1 + 2\beta\partial_{\hat{t}} x_0 + 2\Omega x_0 + x_1 = 0$$
$$\partial_{\hat{t}}^2 x_1 + x_1 = -2\left(i\frac{d}{dT}Ae^{it} - 2i\beta A - 2\Omega Ae^{it}\right) + c.c.$$

Need to avoid secular terms $\Rightarrow$ require

$$\frac{d}{dT}A = -\beta A + i\Omega A$$

then no secular terms arise that grow linearly in time.

Solution

$$\begin{aligned}A &= \mathcal{A}e^{-\beta T + i\Omega T} \\ x_0 &= \mathcal{A}e^{-\beta T}e^{i\hat{t} + i\Omega T} = \mathcal{A}e^{-\epsilon\beta t}e^{i(1+\epsilon\Omega)t}\end{aligned}$$

Thus:

- Two-timing (multiple scales) avoid secular terms and get frequency shift and slow damping correct to the order considered
- calculation easier in complex exponentials

Example: Duffing oscillator

$$\ddot{x} + x + \epsilon x^3 = 0$$

Ansatz:

$$x = x_0(t,T) + \epsilon x_1(t,T) + \cdots$$
$$\left(\frac{d}{dt}\right)^2 \to \partial_t^2 + 2\epsilon\partial_t\partial_T + O(\epsilon^2)$$

$0(\epsilon^0)$:

$$\partial_x^2 x_0 + x_0 = 0 \qquad x_0 = Ae^{it} + A^* e^{-it}$$

$0(\epsilon^1)$:

$$\partial_t^2 x_1 + x_1 + \underbrace{2\partial_t\partial_T x_0}_{2i\frac{dA}{dT}e^{it}+c.c.} + \underbrace{x_0^3}_{A^3e^{3it}+3|A|^2Ae^{it}+3|A|^2A^*e^{-it}+A^{*3}e^{-3it}} = 0$$

thus

$$\frac{d^2}{dt^2}x_1 + x_1 = \underbrace{e^it}_{\text{secular resonance term}} \left\{2i\frac{dA}{dT} + 3|A|^2A\right\} + e^{3it}A^3 + c.c.$$

require:

$$\frac{dA}{dT} = +\frac{3}{2}i|A|^2A \qquad \Rightarrow \qquad A = \mathcal{A}e^{\frac{3}{2}i\mathcal{A}^2t}$$

$$x_0 = \mathcal{A}exp\left(i(1+\frac{3}{2}\epsilon\mathcal{A}^2t\right) + c.c.$$

Notes:

- nonlinearity induces frequency shift

  $\to$ soft and hard spring ($\epsilon \lessgtr 0$)

$$\ddot{x} + (1+\epsilon x^2)x = 0$$

- at higher orders in ϵ additional frequency shifts
  $\Rightarrow$ approximate and exact solution get out of sync for $t \sim O(\frac{1}{\epsilon^2})$

$$\cos\left((\omega + \epsilon\omega_1 + \underbrace{\epsilon^2\omega_2)t}_{\epsilon^2\omega_2 t\sim 2\pi \Rightarrow t=O(\frac{1}{\epsilon^2})}\right)$$

- two-timing also very useful near bifurcation, where one time scale becomes very slow.

More general formulation:

At $O(\epsilon)$ one obtains

$$Lx_1 = I(x_0)$$

with operator L singular: $Lx_0 = 0$

Operators correspond to matrices:

if $\underline{\underline{M}}\,\underline{x}_0 = 0$ then $\det(\underline{\underline{M}}) = 0$ and $\underline{\underline{M}}\,\underline{x} = \underline{b}$ has only a solution for special values of $\underline{b}$:

solvability condition to remove secular terms (see later).

3.4.3 Insertion for HW: Relaxation Oscillations

Class of systems for which one can see the periodic orbit relatively easily: N-shaped nullcline

Consider for $\mu \gg 1$:

$$\begin{aligned}\dot{x} &= \mu(y - F(x))\\ \dot{y} &= x\end{aligned}$$

with $F(x) = -x + x^3$.

`Figure`

For $\mu \gg 1$ horizontal motion much faster than vertical motion, except near the nullcline $y = F(x)$
$\Rightarrow$ slow branch and fast branch on the periodic orbit

`Figure`

Period of periodic orbit determined mostly by time spent on slow branch.

On slow branch

$$\begin{aligned} y \sim F(x) \qquad &\Rightarrow \qquad \dot{y} \sim \frac{dF}{dx}\,\dot{x} \\ &\Rightarrow \qquad \dot{x} = G(x) \end{aligned}$$

$$T = \int dt \sim \int \frac{dt}{dx}\,dx = \int \frac{1}{\dot{x}}\,dx$$

3.4.4 Hopf Bifurcation

Consider example

$$\begin{aligned} \dot{x} &= \mu x - \omega y + \gamma x(x^2+y^2) - \delta y(x^2+y^2) \\ \dot{y} &= \omega x - \mu y + \delta x(x^2+y^2) - \gamma y(x^2+y^2) \end{aligned}$$

Linear stability of $(0,0)$:

eigenvalues of

$$\begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \Rightarrow \lambda_{1,2} = \mu \pm i\omega$$

`Figure`

Rewrite

$$A = x + iy$$

$$\Rightarrow \dot{A} = (\mu + i\omega)A + (\gamma + i\delta)|A|^2 A$$

rewrite again

$$\begin{aligned} A &= Re^{i\Theta} \\ \Rightarrow \dot{R} &= \mu R + \gamma R^3 \\ \dot{\Theta} &= \omega + \delta R^2 \end{aligned}$$

$\Rightarrow$ steady solution

$$
\begin{aligned}
R_0 &= \sqrt{-\frac{\mu}{\gamma}} \qquad \Theta = (\omega + \delta R_0^2)t + \Theta_0 \\
\begin{pmatrix} x \\ y \end{pmatrix} &= R_0 \begin{pmatrix} \cos[(\omega + \delta R_0^2)t + \Theta_0] \\ \sin[(\omega + \delta R_0^2)t + \Theta_0] \end{pmatrix} \qquad \text{periodic orbit}
\end{aligned}
$$

Bifurcation diagrams:

`Figure`

"circles worth of solutions"

Note:

- Solutions exist for any phase Θ_o: continuous family of solutions
- Although this example looks very special, it is the <u>normal form</u> for the Hopf bifurcation and also for weakly nonlinear oscillators, cf. Duffing result

  $$\frac{dA}{dT} = \frac{3}{2}|A|^2 A$$

  corresponds to $\mu = 0, \gamma = 0, \delta = \frac{3}{2}$,
- determinant of linearization does not vanish $\Rightarrow$ via implicit function theorem the number of fixed points does not change.

Example:

$$
\begin{aligned}
\dot{u} &= \mu u - v + u^2 \\
\dot{v} &= u + \mu v + u^2
\end{aligned}
$$

Linear stability of $(0,0)$ again

$$\begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \Rightarrow \lambda = \mu \pm i$$

eigenvectors at the bifurcation point

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \pm i \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

Compare: Duffing oscillator $qquad\ddot{x} + x = -\epsilon x^3$
There we expanded in ϵ: assumed nonlinear term weak

Here: assume u and v small

Based on previous example guess u, $v \sim \mu^{1/2}$

Expansion $\mu = \mu^2\epsilon^2$, $T = \epsilon^2 t$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \epsilon \begin{pmatrix} u_1(t,T) \\ v_1(t,T) \end{pmatrix} + \epsilon^2 \begin{pmatrix} u_2(t,T) \\ v_2(t,T) \end{pmatrix} + \epsilon^3 \begin{pmatrix} u_3(t,T) \\ v_3(t,T) \end{pmatrix} + c.c.$$

insert:

$O(\epsilon)$:

$$\left.\begin{array}{l} \frac{d}{dt}u_1 = -v_1 \\ \frac{d}{dt}v_1 = u_1 \end{array}\right\} \qquad \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \underbrace{A}_{A=A(T)} e^{it} \begin{pmatrix} 1 \\ -i \end{pmatrix} + \underbrace{A^* e^{-it} \begin{pmatrix} 1 \\ +i \end{pmatrix}}_{\text{c.c.}}$$

$O(\epsilon^2)$:

$$\begin{aligned} \frac{d}{dt}u_2 &= -v_2 + u_1^2 \\ \frac{d}{dt}v_2 &= +u_2 + u_1^2 \end{aligned}$$

need u_1^2

$$\begin{aligned} u_1^2 &= e^{2it}A_1^2 + 2|A|^2 + e^{-2it}A_1^{*2} \\ &\Rightarrow \\ u_2 &= B_1 e^{2it} + B_1^* e^{-2it} + C_1 \\ V_2 &= B_2 e^{2it} + B_2^* e^{-2it} + C_2 \end{aligned}$$

$\propto e^{2it}$:

$$\begin{aligned} 2iB_1 &= -B_2 + A_1^2 \\ 2iB_2 &= B_1 + A_1^2 \Rightarrow B_1 = 2iB_2 - A_1^2 \\ -4B_2 - 2iA_1^2 &= -B_2 + A_1^2 \end{aligned}$$

$$B_2 = \frac{1}{3}(-2i - 1)A_1^2$$

$$B_1 = A_1^2(\frac{2}{3}(+2-i)-1) = A_1^2(+\frac{1}{3}-\frac{2}{3}i) = \underline{+\frac{1}{3}(1-2i)A_1^2}$$

$\propto e^{0t}$:

$$\begin{aligned} 0 &= -C_2 + 2|A_1|^2 \qquad C_2 = 2|A_1|^2 \\ 0 &= C_1 + a|A|^2 \qquad \underline{C_1 = -2|A_1|^2} \end{aligned}$$

$O(\epsilon^3)$:

$$\begin{aligned} \partial_T u_1 &+ \partial_t u_3 = \mu_2 u_1 - v_3 + 2u_1u_2 \\ \partial_T v_1 &+ \partial_t v_3 = u_3 + \mu_2 v_1 + 2u_1u_2 \end{aligned}$$

u_1u_2 generates $e^{\pm 3it}$ and $e^{\pm it}$.

reorder:

$$\begin{aligned} \partial_t u_3 + v_3 &= -\partial_T u_1 + \mu_2 u_1 + 2u_1u_2 \equiv I_1 e^{it} + J_1 e^{3it} + c.c. \\ \partial_t v_3 + u_3 &= -\partial_T v_1 + \mu_2 v_1 + 2u_1u_2 \equiv I_2 e^{it} + J_2 e^{3it} + c.c. \end{aligned}$$

$\Rightarrow$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} e^{it} + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} e^{3it} + c.c.$$

need to solve

$$\underbrace{\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}}_{\underline{\underline{M}}_1} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \text{ and } \underbrace{\begin{pmatrix} 3i & 1 \\ -1 & 3i \end{pmatrix}}_{\underline{\underline{M}}_3} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

Now:

- $\det \underline{\underline{M}}_3 \neq 0 \Rightarrow \underline{\underline{M}}_3$ can be inverted $\Rightarrow \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$
- $\det \underline{\underline{M}}_1 = 0 \Rightarrow \underline{\underline{M}}_1$ **cannot be inverted!**

  Solutions exist only if $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ in range of $\underline{\underline{M}}_1$

Determine eigenvectors associated with 0 eigenvalue of $\underline{\underline{M}}_1$:

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ look for left-eigenvector:

$$\begin{aligned}
(u_0^+, v_0^+)\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} &= (0,0) \\
\Rightarrow iu_0^+ - v_0^+ &= 1 \Rightarrow (u_0^+, v_0^+) = (1, +i)
\end{aligned}$$

Multiply equation from the left with (u_0^+, v_x^+)

$$\underbrace{(u_0^+, v_x^+)\underline{\underline{M}}_1}_{0 \text{ for any}\begin{pmatrix} D_1 \\ D_2 \end{pmatrix}} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = (u_0^+, v_0^+)\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$\Rightarrow$ obtain solution only if

$$u_0^+ I_1 + v_0^+ I_2 = 0 \qquad \text{Solvability Condition}$$

Specifically:

$$\begin{aligned}
I_1 &= -\frac{d}{dT}A_1 + \mu_2 A_1 + 2\{-1A_i|A_1|^2 + A_1^*(+\frac{1}{3}(1-2i)A_1^2\} \\
I_2 &= -\frac{d}{dT}A_1(-i) + \mu_2(-i) + 2\{-2A_1|A_1|^2 + A_1^*(+\frac{1}{3}(1-2i)A_1^2\}
\end{aligned}$$

$$\frac{d}{dT}A_1 = \mu_2 A_1 - |A_1|^2 A_1 \left\{+1 + \frac{7}{3}i\right\}$$

Notes:

- The solvability condition arises because linearization around fixed point is singular: always the case for bifurcation problems (steady bifurcation see later)
- Fredholm alternative: either $\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$ satisfies solvability condition or there is no solution.

Origin of normal form: time translation symmetry

look at periodic solution

$$U(t) = e^{it}A(T) = e^{it}R_0 e^{i\frac{7}{3}R_0^2 T + i\Theta_0}$$

coefficients in system are time-independent

$$\Rightarrow \tilde{u}(t) \equiv e^{i(t-t_0)} R_0 e^{i\frac{7}{3}R_0^2 T + i\Theta_0}$$

is also a solution

$\tilde{u}(t)$ can also be written as

$$\tilde{u}(t) = e^{it} R_0 e^{i\frac{7}{3}R_0 T + i\underbrace{(\Theta_0 - t_0)}_{\text{shift in } \Theta}}$$

$\Rightarrow A(T)e^{i\Theta_0}$ must be a solution for any Θ_0.

$\Rightarrow$ the only nonlinear terms that are allowed are $|A_1|^{2n} A_1$.