Interdisciplinary Nonlinear Dynamics (438)

Fall 2002
Hermann Riecke

Take-Home Final
Due December 12, noon

1. **One-Dimensional Flows**

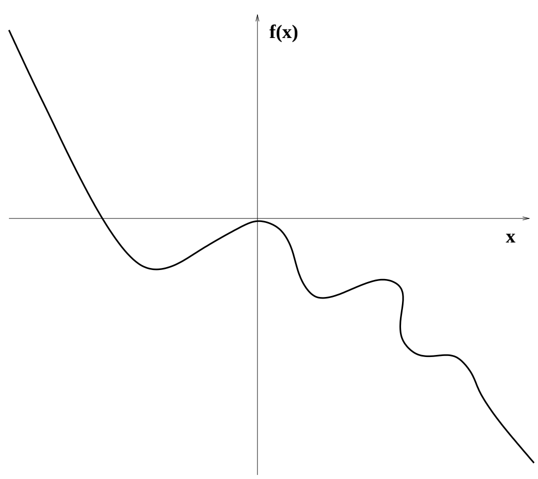
   Consider the one-dimensional system

   $$\dot{x} = f(x, \mu). \qquad (1)$$

   As sketched below, for $\mu = 0$ the function $f(x, \mu)$ has a horizontal tangent at the origin. Assume that varying μ rotates the graph of $f(x, \mu)$ around the origin. Assume that $f \propto x$ for large x.

   (a) What kind of bifurcations does the system undergo as μ is varied?

   (b) Sketch the bifurcation diagram, marking the stability of the solutions on the various branches, i.e. as a function of μ sketch all solution branches, mark the bifurcation points and indicate the type of bifurcation.



2. **Center-Manifold Reduction**

   Perform a center-manifold reduction for the system

   $$\dot{x} = \alpha x + \gamma \sin(xy), \qquad (2)$$
   $$\dot{y} = -y + 1 - \cos x. \qquad (3)$$

   Thus, find a simple fixed point, determine its linear stability, and identify the stable, center, and unstable eigenspaces, respectively. Then determine the center manifold and the dynamics on it near the fixed point. Keep only the leading order terms in the expansion. Use the suspended system, to allow also a slight variation of the control parameter α. What kind of bifurcations do you obtain?

3. **Tricritical Bifurcation in a Toy-Model for Taylor Vortex Flow**

   In systems that allow varying two parameters rather than one often a change from a supercritical to a subcritical bifurcation can be observed. As an example you may want to take a look at the experiments on Taylor vortex flow discussed in [1]. You get that paper easily at `http://ojps.aip.org/prlo`. We model similar (albeit slightly simpler) behavior with a modified Swift-Hohenberg model,

$$\partial_t \psi = R\psi - \left(\partial_x^2 + 1\right)^2 \psi + \alpha\psi^3 - \psi^5, \qquad -\infty < x < \infty. \tag{4}$$

(a) For $\alpha = 0$ perform a weakly nonlinear analysis of (4) using suitably scaled slow space and time variables to derive a Ginzburg-Landau equation for the amplitude of the spatial pattern.

(b) Now include the cubic term proportional to α in the weakly nonlinear analysis. How do you have to scale α to obtain a single Ginzburg-Landau equation that reflects both the cubic and the quintic term of (4)?

(c) For appropriately chosen values of α, sketch the various qualitatively different bifurcation diagrams you obtain when varying R.

4. **Periodically Forced Pendulum**

Consider the periodically forced, weakly damped pendulum described by

$$\frac{d^2y}{dt^2} + (1 + \Delta\omega)^2 \sin y + \gamma\frac{dy}{dt} + \alpha\left(\frac{dy}{dt}\right)^3 = 2f\cos(\Omega t). \tag{5}$$

(a) In the absence of forcing, $f = 0$, without linear damping, $\gamma = 0$, and without detuning, $\Delta\omega = 0$, perform a weakly nonlinear analysis of the pendulum, i.e. for small amplitudes use multiple time scales to derive a differential equation ('amplitude equation') for the slow evolution of the amplitude of the pendulum. What would happen in your expansion for y if the amplitude did not satisfy the amplitude equation?

(b) Now introduce linear damping, forcing, and detuning. Consider the case $\Omega = 1$. In order for these effects to appear simply as additional terms in the amplitude equation they should be weak. How do you have to scale γ, f, and $\Delta\omega$ to achieve this? With this scaling include all three terms in the analysis and derive an amplitude equation that includes linear damping, forcing, and detuning.

(c) Find simple, stationary solutions for the amplitude by writing it in terms of its magnitude and phase, $A = R\,exp(i\phi)$. Eliminating the phase you get a bicubic equation for the magnitude R. Investigate its solutions by solving for the detuning as a function of R. Plot $\Delta\omega(R)$ for representative cases. To do so, choose $\alpha = 0.02$ and $\gamma = 0.05$ fixed and vary f. Interpret your results: what do the stationary solutions for A represent physically? Does the system undergo bifurcations when $\Delta\omega$ is varied? What type are they? Is there hysteresis?

(d) Solve (5) numerically by writing it as a system of 2 first-order equations. With $\alpha = 0.05$, and $\gamma = 0.001$ fixed, investigate two regimes:

   i. Weak forcing: take $f = 0.1$ (and keep $\Omega = 1$) and study the behavior of the pendulum as you vary the detuning $\Delta\omega$. Near $\Delta\omega = 0$, do you find solutions that agree qualitatively with your analytic results? What happens when $|\Delta\omega|$ becomes larger?

   ii. Strong forcing: now switch to lower frequencies and take $\Omega = 0.15$. Scan the behavior of the system as you increase the forcing in steps ($\Delta f \approx 0.01$) from $f = 0.27$ to $f = 0.30$. Do you find qualitatively different solutions depending on whether you take large or small initial velocities (you may have to run to times $t = \mathcal{O}(1000)$ to get an idea of the solutions)? Print representative plots of the solutions.

References

[1] A. Aitta, G. Ahlers, and D. S. Cannell. Tricritical phenomena in rotating Couette-Taylor flow. *Phys. Rev. Lett.*, 54:673, 1985.