

# Interdisciplinary Nonlinear Dynamics (438)

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## Problem Set 1

For Discussion Section October 16

**Note:** the discussion section is on Wednesdays 5-6 in L251

### 1. Some Short Nice Problems from Strogatz

2.2.9, 2.2.10.

### 2. Population Growth with Eggs

Consider the dynamics of a population of animals that lay eggs and investigate the following simple model. The animals propagate by laying eggs, which only after a time  $\tau$  have hatched and matured and can lay new eggs. The rate at which they lay eggs is adversely affected if the population  $N$  is too dense, i.e. assume the rate decreases linearly in  $N$ . Similarly the death rate increases linearly with  $N$ . Thus, you get

$$\frac{dN}{dt} = -aN(t) - bN(t)^2 + \alpha N(t - \tau) - \beta N(t - \tau)^2. \quad (1)$$

- Nondimensionalize the evolution equation by using the magnitude of the death rates  $|a|$  and  $|b|$  as characteristic scales (in order to leave the delay as a control parameter). In this process  $\alpha$ ,  $\beta$ , and  $\tau$  will be rescaled to  $\tilde{\alpha}$ ,  $\tilde{\beta}$ , and  $\tilde{\tau}$ . In the rest of the problem we will omit the  $\tilde{\phantom{x}}$  over the symbols again and write  $\alpha$ , etc.
- Use the matlab program available on the class web site to solve first the usual logistic equation, in which the delay vanishes,  $\tau = 0$ , and  $a = -1$  and  $b = 1$ . Compare the numerical to the analytical solution by calculating

$$E^2 = \frac{\int_0^{t_{max}} (n_{ex}(t) - n_{num}(t))^2 dt}{\int_0^{t_{max}} (n_{ex}(t))^2 dt}, \quad (2)$$

where  $n$  is the dimensionless density and  $t_{max}$  is chosen appropriately so that  $n$  has just about reached its stationary value. Measure the error  $E$  as a function of the size  $\Delta t$  of the time step, decreasing  $\Delta t$  repeatedly by factors of 2. Plot  $E(\Delta t)$  double-logarithmically<sup>1</sup> and confirm that for small  $\Delta t$  it is linear in  $\Delta t$ . Choose the time step small enough to get the error below 1%.

- To get a feeling for the *stability* of the simple forward Euler method used in the matlab program investigate the behavior of the numerical solution as you increase  $\Delta t$  in factors of 2 up to values of  $\Delta t = 3$ . What change in the behavior do you observe?
- Modify the matlab program to solve the (dimensionless) evolution equation for  $\tau \neq 0$ . Now choose  $a = +1$  and  $b = 1$ . The initial conditions require the knowledge of  $n(t)$  for  $-\tau < t < 0$ . Use  $n(t) = 0$  for that range of  $t$  (no eggs laid yet).
- Study the dynamics of the equation numerically for  $\tau = 3$  and  $\beta = 3.7$  over a range of  $\alpha$  between 10 and 20 (all in dimensionless quantities). Does the behavior change qualitatively when changing  $\alpha$ ? How does it differ from that of the simple logistic equation without delay? How is this behavior possible although the equation is only first-order in time?

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<sup>1</sup>Use the matlab command `loglog` instead of `plot`.

- (f) Establish that the behavior you observe is not a numerical artifact by performing a convergence test for one representative case. Since you do not know the exact solution plot the difference  $n(t_0, \Delta t) - n(t_0, \Delta t/2)$  for some suitably chosen fixed  $t_0$  double-logarithmically as a function of  $\Delta t$  and establish that this difference shows the correct scaling in  $\Delta t$ .

### 3. Uniqueness of Solutions

- (a) Consider

$$\frac{du}{dt} = f(u). \quad (3)$$

with  $f(u_0) = 0$ .

- i. Show that for functions  $f(u)$  that satisfy the Lipschitz condition in an interval  $[u_0 - \Delta u, u_0 + \Delta u]$  around the stable fixed point  $u_0$  the time to reach that fixed point is infinite. Discuss this result in view of the uniqueness theorem for solutions stated in class.
- ii. In class it was stated that if  $f'(u)$  is continuous the solution to (3) is unique. Discuss in this context the equation

$$\frac{du}{dt} = |u|, \quad (4)$$

for which clearly  $f'(u)$  is not continuous. Is the solution to this equation unique for all initial conditions? Why not? Why?

- (b) Do Strogatz 2.5.6 (leaky bucket problem).

### 4. Predictor-Corrector Methods

Consider the differential equation

$$\frac{du}{dt} = f(u). \quad (5)$$

The solution  $u_{j+1} \equiv u(t + \Delta t)$  can be written exactly as

$$u_{j+1} = u_j + \int_{t_j}^{t_{j+1}} f(u(t)) dt, \quad (6)$$

where  $t_j = j \Delta t$ . Use the trapezoidal method to approximate the integral in (6). This would require the knowledge of  $u_{j+1}$ , which is still unknown. Determine an approximation  $\bar{u}$  for  $u_{j+1}$  in the trapezoidal rule by a forward Euler step starting from  $u_j$  (this step is called the *predictor step*) and use it to determine then  $u_{j+1}$  (this step is called the *corrector step*). What order in  $\Delta t$  is the error for this *predictor-corrector* scheme?