Interdisciplinary Nonlinear Dynamics (438)
Fall 2002
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Problem Set 4

For Discussion Section November 6

1. **Poincaré-Bendixson Theorem**

   Consider the nonlinear system

$$\frac{dx}{dt} = y + x^3 - x\,(x^2+y^2)^2, \tag{1}$$
$$\frac{dy}{dt} = -x + y^3 - y\,(x^2+y^2)^2. \tag{2}$$

   (a) Perform a linear stability analysis of the fixed point $(0,0)$ and use the Poincaré-Bendixson Theorem to show that (1,2) has a limit cycle.

   (b) Solve (1,2) numerically[1] to check your result from a). Plot representative orbits to demonstrate your result. Is the orbit stable or unstable?

2. **Damped Driven Pendulum**

   Consider a pendulum with friction that is exposed to a constant torque Γ,

$$ml^2\frac{d^2\theta}{dt^2} + \beta\frac{d\theta}{dt} = -mgl\sin\theta + \Gamma. \tag{3}$$

   (a) Rewrite (3) as a system of first-order equations and determine all fixed points of this system.

   (b) Perform a linear stability analysis of all the fixed points. Discuss what happens physically when the fixed points become unstable! Can you use this to make an educated guess as to what kind of bifurcation the instability leads to?

3. **Relaxation Oscillations**

   Read Ch.7.5 in Strogatz and do problem 7.5.4. Solve also the equations numerically and measure (roughly) the period of the oscillation. Compare the numerically determined period with that from the analytical approximation obtained for $\mu \gg 1$.

4. **Regular Perturbation Theory**

   Consider the two problems

$$\text{a)}\qquad \ddot{x} + x = \epsilon \tag{4}$$
$$\text{b)}\qquad \ddot{x} + x = \epsilon\ddot{x} \tag{5}$$

   with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$. For both cases:

   (a) solve the problem exactly.

   (b) perform a regular perturbation expansion

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \mathcal{O}(\epsilon^3). \tag{6}$$

   Does the expansion lead to secular terms? Why? Why not?

[1] You may want to go from the Euler scheme used in the previous problems to a 4th-order Runge-Kutta scheme.