Interdisciplinary Nonlinear Dynamics (438)

Fall 2001

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Problem Set 5

For Discussion Section November 13

1. **Weakly Nonlinear Analysis of Hopf Bifurcation**

   Consider the two coupled equations

$$\frac{dx}{dt} = \mu x + y + ax^3, \tag{1}$$

$$\frac{dy}{dt} = -x + \mu y + by^3. \tag{2}$$

   (a) Sketch the nullclines of (1,2) and use them to determine qualitatively the flow as much as that is possible from that information.

   (b) Perform a linear stability analysis of the fixed point $(0,0)$ including the eigenvectors associated with the relevant eigenvalues.

   (c) Solve (1,2) numerically (write your own code or use `pplane.m`[1]) and compare with the weakly nonlinear analysis. Specifically, measure (roughly) the amplitude of the oscillations and their frequency for a suitable range of the control parameter μ.

   (d) Use the eigenvectors as the first-order terms in a weakly nonlinear analysis of the Hopf bifurcation and use multiple time scales to obtain an evolution equation for the slowly varying complex amplitude of the oscillation. Depending on a and b, can the bifurcation be supercritical and/or subcritical?

2. **An Interesting Dynamical System**[2]

   Consider the dynamical system

$$\frac{dx}{dt} = y, \tag{3}$$

$$\frac{dy}{dt} = \alpha x + \beta y + ax^3 - x^2 y. \tag{4}$$

   In this problem you will study this system analytically and numerically. For the numerical simulations you can use `pplane5`[3] or write your own code. `pplane` has the advantage to provide you right away with a number of graphical options.

   (a) Determine analytically the stability of the fixed point $(x, y) = (0, 0)$ as a function of α and β. Based on this information, what type of behaviors may be expected as α and/or β is varied?

   (b) Consider the case $a = -1$.

      i. Solve (3,4) numerically. What kind of nonlinear behavior do you find if you vary α across the instability identified above for $\beta = -1$?
      For $\alpha = 0$ and $\beta = -1$ identify the center manifold of the attractor numerically by starting from a number of different initial conditions. Increase now α to 0.05, 0.1, and 1 and describe what then happens to the center manifold?

[1] See the class web site for downloading that software.

[2] This is one of the interesting situations that can arise at a codimension-2 bifurcation, i.e. when two modes go unstable at the same parameter value (cf. Guckenheimer & Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Ch. 7.3)

[3] You may want to set the solver to `ODE23` and set the calculation window to 10 times the display window.

ii. For small α perform analytically a center-manifold reduction of (3,4), i.e. determine a center manifold that describes the dynamics of this system for small amplitudes x, y, and for small values of the bifurcation parameter α. Go to cubic order. Obtain the evolution equation for the dynamics of the system on that manifold. Compare the analytically obtained center manifold and the fixed points on it quantitatively with your numerical results.