D-21-1 Models in Applied Mathematics
Fall 1997
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Problem Set 1 October 2

Due October 14, 1997

1. Calculate systematically the solutions to the equation describing the forced frictionless harmonic oscillator. In the non-resonant as well as the resonant case, use variation of parameters to find the solution that satisfies the initial conditions $x(t=0)=0$ and $\dot{x}(t=0)=0$. In the almost resonant case ($\omega-\omega_0$ small) show that the general solution exhibits 'beating', i.e. the amplitude of the fast oscillations exhibit a slow variation on the time scale $O(1/(\omega-\omega_0))$.

2. The following problem is phrased vaguely on purpose. When confronted with the task to model a concrete system it is often not clear right away which features of the system are important and which are not. It is an art to strike a balance between relevance and simplicity of the model.
a) Make a simple model of a shock absorber for a car and of a "test road", i.e. of a time-dependent force acting on the absorber which mimics some aspect of a road which you find important. Choose the model in a way that you can calculate the motion analytically (you could use Maple if you want to) and that at the same time the model retains relevant features.
b) Calculate the solution for suitable initial conditions and plot the vertical position of the car, its vertical acceleration, and the extension (compression) of the shock absorber as a function of time[1]. Do so for a number of qualitatively different situations you find relevant.
c) Give the condition that the wheel is not loosing contact with the road at any time.
d) Discuss obvious short-comings of your model.
After setting up a model and finding appropriate solutions engineers are confronted with the question of the proper design, i.e. how should the parameters be chosen in order to optimize the performance? Can you make general remarks how the stiffness and the damping of the spring should be chosen if an optimally smooth ride (minimal vertical acceleration) is to be achieved? Are there aspects because of which one may have to settle for sub-optimal parameters? Keep in mind that the mass and the size of the vehicle are essentially fixed and that the car sometimes has to go around corners quickly as well as through pot holes and over large bumps and hills. Thus the design would be different for a sports and for a luxury family car.

3. Consider a pendulum without friction. Give an exact expression for its period. To do so, show that a first integral of
$$\theta'' + \omega_0^2 \sin\theta = 0 \tag{1}$$
is given by
$$\theta'^2 = 4\omega_0^2\left(\sin^2 a/2 - \sin^2\theta/2\right) \tag{2}$$
where A is the maximum amplitude. Verify that the transformation
$$\sin\theta/2 = \sin a/2\,\sin\psi \tag{3}$$

[1]The plotting is done most easily with Maple, Mathematica of Matlab. Maple and Matlab are available on the HP-lab in the basement of Tech. If you don't have access to any computers: it's easy to get an account for the HP-lab.

brings the solution into the form

$$t = -\omega_0^{-1} \int_{\pi/2}^{\psi} (1 - k^2 \sin^2 \psi)^{-1/2} d\psi \tag{4}$$

with $k^2 = \sin^2 a/2$. Give a perturbation expansion for the period in terms of small amplitudes. Calculate the first two non-vanishing terms. Compare the result to that of a harmonic oscillator (cf. problem 9 in Ch.2.2 of Lin & Segel).