D-21-1 Models in Applied Mathematics
Fall 1997
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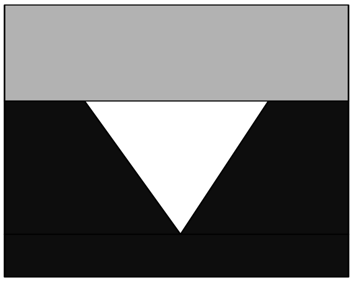
Problem Set 4 November 24, 1997

Due December 11, 1997

1. Consider a material with density ρ, specific heat c_v, and heat conductivity k. Assume that in this material a exothermal chemical reaction takes place which releases heat at a rate and density of $Q(\mathbf{r})$. Use the first law of thermodynamics to derive an evolution equation for the temperature $T(\mathbf{r}, t)$ in this material.

2. Consider a plate of thickness d which is infinitely extended in the other two directions. Assume it is made of a material with very small heat diffusivity κ. Let the temperature at its bottom be $T_l(x, y)$ and at its top T_u. Consider the limit $\kappa \to 0$. Show that for suitable choice of T_u the heat flux q through the plate is independent of x and y and perpendicular to the plate to leading order in κ. To do so, consider an expansion in κ of T_u and of q.



3. Consider a conical body described by $x^2 + y^2 = z^2$ for $0 \leq z \leq 1$ made from the reactive material of problem 1) (with $Q = Q_0 = const$) and embedded in two different materials as shown in the figure. At the perimeter of the whole set-up the black material is kept at a temperature T_b while the grey material is kept at a temperature T_g.
a) Use the results obtained in problem 2) to argue that the heat flux across the top surface of the cone is independent of the space-dependence of the surface temperature in the limit of vanishing heat diffusivity in the grey material.
b) How does the diffusivity of the black material have to be chosen in order to approximate a fixed-temperature boundary condition at the interface between the cone and the black material?
c) Show that a steady temperature field of the form $T = a + bx^2 + cy^2 + dz^2$ satisfies the relevant heat equation inside the cone and the boundary conditions of fixed flux across the top surface of the cone and a fixed temperature at the other surfaces. Determine a, b, c, and d. Determine the heat flux across the top surface of the cone.
d) Calculate the total heat flux out of the cone directly *via* the surface integral.
e) Verify that the heat flux determined in d) agrees with the rate at which heat is produced in the cone.

Note: If you use cylindrical coordinates you might find the following expressions useful:

$$\nabla U = \partial_r U \underline{e}_r + \frac{1}{r}\partial_\phi U \underline{e}_\phi + \partial_z U \underline{e}_z,$$

$$\Delta U = \frac{1}{r}\partial_r\{r\partial_r U\} + \frac{1}{r^2}\partial_\phi^2 U + \partial_z^2 U,$$

$$\nabla \cdot \underline{v} = \frac{1}{r}\partial_r\{r v_r\} + \frac{1}{r}\partial_\phi v_\phi + \partial_z v_z.$$

4. Consider an infinite, homogeneous bar.
   a) Determine the temperature profile in the bar as a function of time if initially the temperature is given by $T(x) = T_0\, H(x)$, with $H(x)$ the Heaviside step function.
   b) If the bar had a finite length L, can you give an estimate for the time τ_L after which you would expect the temperature profile to differ appreciably from that determined in a) if the initial jump in temperature occurred in the middle of the bar? Of particular interest is the scaling of τ_L with the various parameters in the problem.
   Note: The error function $\mathrm{erf}(x)$ is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\, dz.$$

5. Consider a thin cooling fin of a computer chip or a radiator. For simplicity assume it is two-dimensional - or infinitely extended in the third dimension - and assume that the fin is so thin in its transverse direction (y) that the temperature can be assumed to depend only on the direction along the fin (x). Derive a partial differential equation for the temperature in the cooling fin that takes into account the heat transfer to the ambient air and allows for the possibility that the thickness of the fin varies in some general fashion (e.g. tapers out towards its end). Assume that there is a fan that blows the air across the fin.