ESAM 446-1
Numerical Solution of Partial Differential Equations

Fall 2004

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Problem Set 2

This problem set is due October 18 in class.

1. Derive the 6^{th} order central difference approximation of d^2u/dx^2, i.e. the error will be of $O(\Delta x^6)$ using i) Taylor expansion of $u(x+h)$ and ii) Fourier analysis. Find the Fourier representation of the derivative obtained in this approximation and the leading-order term in the relative error.

2. Derive the second-order approximation of du/dx using forward differences, i.e. involving only points to the right of the point at which the derivative is to be approximated. Do a Fourier analysis of this approximation and find again the leading-order term in the relative error. Separate it into real and imaginary part and discuss the effect of the two terms if this scheme is used in the PDE $\partial_t u = \partial_x u$.

3. Use the result of problem 2 to find the second-order backward-difference approximation of du/dx. **Do not derive it from scratch!**

4. Show that every central-difference approximation for du/dx vanishes at $kh = \pi$.

5. Consider the initial-value problem

   Have them also change Δx: one-sided FE growth rate diverges like $1/\Delta x$ while growth rate for central FE goes to 0 for $\Delta t \to 0$.

$$\partial_t u = -\partial_x u, \qquad u(x,0) = \cos(2\pi x), \qquad 0 \leq x \leq 10 \tag{1}$$

   with periodic boundary conditions.

   (a) Find the exact solution.

   (b) Compute the numerical solution for $t = 2$ using the following schemes and parameters[1]:

      i. Forward Euler with central differences, $\Delta x = 0.01$, and a sequence of Δt, $\Delta t = 0.01, 0.005, 0.0025$.

      ii. Forward Euler with forward differences in space, $\Delta x = 0.01$ and a sequence of Δt, $\Delta t = 0.01, 0.011, 0.005$.

      iii. Forward Euler with backward differences in space, $\Delta x = 0.01$ and a sequence of Δt, $\Delta t = 0.01, 0.011, 0.005$.

      iv. Lax-Friedrich with $\Delta x = 0.01$ and a sequence of Δt, $\Delta t = 0.01, 0.013, 0.005$.

[1]**Note:** On the class web site is a matlab template for a program that also shows a movie of the simulation on the class web site. That code is so short that I decided not to divide it into m-files.

For each scheme describe the results in a few words and show one representative plot of the solution. How do the results compare to your expectations based on Neumann analysis?

P.S. Since this homework involves a lot of equations, a hand-written solution is just fine.