446-1
Numerical Solution of Partial Differential Equations
Fall 2004
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Problem Set 3 (due Friday, November 5)

Consider the 2-way wave equation

$$\partial_t u = \frac{1}{\rho(x)}\partial_x v, \qquad \partial_t v = \rho(x)c(x)^2\partial_x u \tag{1}$$

describing waves, e.g. sound waves, in a one-dimensional medium with space-dependent density and elastic constant (compressibility).

1. **Neumann Analysis of (2,2)-MacCormack**
   Perform a Neumann analysis of the (2,2)-MacCormack scheme for the two-way wave equation (1) with ρ and c independent of x. What is the stability limit of this scheme? Note that the easiest way to treat coupled PDE is to consider the vector $\mathbf{w} = (u, v)$ and write the equations as a matrix differential equation. How many solutions z do you obtain? How are they related? How many different values of $|z|$ are there?

2. **Various Ways for the Two-Way Wave Equation**

   Solve (1) numerically with periodic boundary conditions on the domain $[0, 1]$. Use as initial condition a pulse of width Δ

   $$u(x,0) = Ae^{-\frac{(x-0.5)^2}{2\Delta^2}}, \qquad v(x,0) = Ac\rho e^{-\frac{(x-0.5)^2}{2\Delta^2}} \tag{2}$$

   with $A = 0.3$. Let the solution evolve from $t = 0$ to $t = 1$.

   (a) Implement the (2,2)-MacCormack and the (4,4)-Runge-Kutta scheme for (1). Note that in a subsequent problem the density $\rho(x)$ and the velocity $c(x)$ will be allowed to depend on the position x. Therefore structure your code in a way that allows c and ρ to be arbitrary functions of space.

   (b) Validate each of the two schemes for $\rho = 2$, $c = 1$, and $\Delta = 0.03$ by calculating the error at $t = 1$

   $$E_2^2 = \frac{\sum_{j=1}^{N}(u_j^n - u_e(x_j, t=1))^2}{\sum_{j=1}^{N} u_e(x_j, t=1)^2}. \tag{3}$$

   Here u_j^n stands for the numerical solution at $t = n \cdot \Delta t \equiv 1$ and position $x = j \cdot \Delta x$. In an infinite system (no periodic boundary conditions) the exact solution $\mathbf{w}_e(x,t) = (u_e(x,t), v_e(x,t))$ can be shown to be given by

   $$u_e(x,t) = Ae^{-\frac{(x+ct-0.5)^2}{2\Delta^2}}, \qquad v_e(x,t) = Ac\rho e^{-\frac{(x+ct-0.5)^2}{2\Delta^2}}. \tag{4}$$

   Demonstrate the correct convergence of your code by plotting the error double-logarithmically as a function of Δx for a suitable range of Δx. Keep λ fixed and close to the stability limit.

(c) For each scheme, show three graphs of the solution (at $t = 1$) for suitable values of Δx to illustrate the convergence. Judged from your simulations what is the sign of the phase error? For easier comparison of the solutions overlay them in the same plot.