D-46-2
Numerical Solution of Partial Differential Equations
Winter 1997
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Problem Set 1, January 16, 1997

due January 23, 1997

1. Projections

Show that the spectral projection operator $P_N\underline{u} \equiv \sum_{|k|=0}^{N} u_k\underline{k}$ is indeed a projection operator, i.e. $P_N^2 = P_N$.

2. Continuous Fourier Transformation

Calculate the Fourier transform of
a) $f_1(x) = x - \pi$ on $0 \leq x \leq 2\pi$,
b) $f_2(x) = x^2 + \theta(x-\pi)((x-2\pi)^2 - x^2)$ on $0 \leq x \leq 2\pi$,
by hand ($\theta(x)$ denotes the Heaviside function). Plot the approximation $P_N f_i(x)$ together with the original function $f_i(x)$ for representative values of N. In each case discuss where the main source of the error lies. Discuss the Gibbs phenomenon.

Calculate the Fourier transform of
c) $f_3(x) = \cos x/(5 - 4\cos x)$ on $0 \leq x \leq 2\pi$,
d) $f_4(x) = e^{-0.5(x-\pi)^2}$ on $0 \leq x \leq 2\pi$.
using Maple (see sample program). Calculate the error of successive approximations (increasing N up to $N = 32$) in the L_2-norm, $||u||^2 \equiv \int_0^{2\pi} u^2(x)\, dx$.
Plot the Fourier coefficients as a function of k. Compare their behavior with the general arguments about the decay of the mode amplitudes (use both double-logarithmic and linear-logarithmic plots). Plot the approximation $P_N f_i(x)$ together with the original function $f_i(x)$ for representative values of N. Plot the error as a function of N (again use both double-logarithmic and linear-logarithmic plots).

3. Discrete Fourier Transformation

Calculate the discrete Fourier transform of
a) $f_1(x) = x - \pi$
b) $f_3(x) = \cos x/(5 - 4\cos x)$
for a range of the number of grid points ($N = 4, 8, 12, 16, 32$). Compare the coefficients obtained by the DFT for small and large values of N. Should they have a given coefficient have the same value independent of N? Compare the values from the DFT with those from the CFT in problem **1.a,c**. For these comparisons plot the coefficients together in a linear-logarithmic plot. What is the origin of the difference between the two sets of coefficients (if there is any)?

4. Fast Fourier Transformation

Calculate the discrete Fourier transform of
a) $f_5(x) = \sin(x) + 4\cos(x)$ (test case).
b) $f_1(x) = x - \pi$
c) $f_3(x) = \cos x/(5 - 4\cos x)$
using an FFT-routine (see e-mail) for $N = 32$. Compare the result with that of the discrete Fourier transform. Check your FFT also by calculating the inverse FFT of the FFT-results in a)-c) and comparing with the original function, i.e. calculate the L_2-error,

$$E_2^2 \equiv \int_0^{2\pi} \left(f(x) - f_{approx}(x)\right)^2 \, dx.$$

Since the discrete Fourier transform relies on the grid of N points, use those N points to calculate the integral. Is the error you get compatible with your expectation of the numerical round-off error?