Spatio-Temporal Chaos
and Defects

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Spatio-Temporal Chaos

Spiral-Defect Chaos
(Morris, Bodenschatz, Cannell, Ahlers, 1993)

Undulation Chaos
(Daniels and Bodenschatz, 2002)
Dislocations in Stripe-Based Spatio-temporal Chaos

- statistics for number of dislocations
  - complex Ginzburg-Landau equation
    (Gil, Lega, and Meunier, 1990)
  - electroconvection
    (Rehberg et al., 1989)
  - undulation chaos
    (Daniels and Bodenschatz, 2002)

- reconstruction of pattern from dislocations
  (La Porta and Surko, 1999)

- dynamical dimension of defects
  - complex Ginzburg-Landau equation
    (Egolf, 1998)
  - excitable media
    (Strain and Greenside, 1998)

Here:

- **hexagonal patterns**: penta-hepta defects

- **stripe-based patterns**: statistics of trajectories of dislocations
Non-Boussinesq Convection

(Bodenschatz, de Bruyn, Ahlers, Cannell, 1991)
Ginzburg-Landau Equations for Hexagons

- **Envelope equations:**
  Slow variation of wavevector and magnitude

\[
v(\tilde{r}, \tilde{t}) = \varepsilon \sum_{j=1}^{3} A_j(\varepsilon \tilde{r}, \varepsilon^2 \tilde{t}) e^{i\tilde{q}_j \cdot \tilde{r}} f(z) + c.c. + h.o.t.
\]

\[
\partial_t A_1 = \mu A_1 + (n_1 \cdot \nabla)^2 A_1 + A^*_2 A^*_3 - A_1 |A_1|^2 \\
- \nu A_1 |A_2|^2 - \nu A_1 |A_3|^2
\]

- **Strong resonance** \(A^*_2 A^*_3\)
- **Variational system**
Ginzburg-Landau Equations for Hexagons

- **Envelope equations:**
  Slow variation of wavevector and magnitude

\[ v(\tilde{r}, \tilde{t}) = \varepsilon \sum_{j=1}^{3} A_j(\varepsilon \tilde{r}, \varepsilon^2 \tilde{t}) e^{i\tilde{q}_j \cdot \tilde{r}} f(z) + c.c. + h.o.t. \]

\[ \partial_t A_1 = \mu A_1 + (n_1 \cdot \nabla)^2 A_1 + A_2^* A_3^* - A_1 |A_1|^2 \]
\[ - \nu A_1 |A_2|^2 - \nu A_1 |A_3|^2 \]
\[ + i \alpha_1 A_3^* (n_2 \cdot \nabla) A_2^* + i \alpha_1 A_2^* (n_3 \cdot \nabla) A_3^* \]
\[ + i \alpha_2 (A_3^* (\tau_2 \cdot \nabla) A_2^* - A_2^* (\tau_3 \cdot \nabla) A_3^*) \]

- Strong resonance \( A_2^* A_3^* \)
- Nonlinear gradient terms
Ginzburg-Landau Equations for Hexagons

- **Envelope equations:**
  Slow variation of wavevector and magnitude

\[
v(\tilde{r}, \tilde{t}) = \varepsilon \sum_{j=1}^{3} A_j(\varepsilon \tilde{r}, \varepsilon^2 \tilde{t}) e^{i\vec{q}_j \cdot \vec{r}} f(z) + c.c. + h.o.t.
\]

\[
\partial_t A_1 = \mu A_1 + (n_1 \cdot \nabla)^2 A_1 + A_2^* A_3^* - A_1 |A_1|^2 \\
- (\nu + \gamma) A_1 |A_2|^2 - (\nu - \gamma) A_1 |A_3|^2 \\
+ i(\alpha_1 - \alpha_3) A_3^* (n_2 \cdot \nabla) A_2^* + i(\alpha_1 + \alpha_3) A_2^* (n_3 \cdot \nabla) A_3^* \\
+ i\alpha_2 (A_3^* (\tau_2 \cdot \nabla) A_2^* - A_2^* (\tau_3 \cdot \nabla) A_3^*)
\]

- **Strong resonance** $A_2^* A_3^*$
- **Rotation**
Ginzburg-Landau Equations for Hexagons

- **Envelope equations:**
  Slow variation of wavevector and magnitude

\[
v(\tilde{r}, \tilde{t}) = \varepsilon \sum_{j=1}^{3} A_j(\varepsilon \tilde{r}, \varepsilon^2 \tilde{t}) e^{i\tilde{q}_j \cdot \tilde{r}} f(z) + c.c. + h.o.t.
\]

\[
\partial_t A_1 = \mu A_1 + (n_1 \cdot \nabla)^2 A_1 + A_2^* A_3^* - A_1 |A_1|^2 \\
- (v + \gamma)A_1 |A_2|^2 - (v - \gamma) A_1 |A_3|^2 - i\beta A_1 (\hat{t}_1 \cdot \nabla)\mathcal{Q} \\
+ i(\alpha_1 - \alpha_3) A_3^* (n_2 \cdot \nabla) A_2^* + i(\alpha_1 + \alpha_3) A_2^* (n_3 \cdot \nabla) A_3^* \\
+ i\alpha_2 (A_3^* (\tau_2 \cdot \nabla) A_2^* - A_2^* (\tau_3 \cdot \nabla) A_3^*)
\]

\[
\nabla^2 \mathcal{Q} = \sum_{j=1}^{3} \left[ 2(\hat{n}_j \cdot \nabla)(\hat{t}_j \cdot \nabla) + \tau \left( (\hat{n}_j \cdot \nabla)^2 - (\hat{t}_j \cdot \nabla)^2 \right) \right] |A_j|^2
\]

with drift velocity \( V \propto \nabla \times (Q\hat{e}_z) \)

- Strong resonance \( A_2^* A_3^* \)
- Rotation
- Mean flow
Penta-Hepta Defects

- Defects in individual roll systems: dislocations
- Resonant term $A_{j-1}^* A_{j+1}^*$: strong attraction between two dislocations in different roll systems $\Rightarrow$ penta-hepta defect

(Eckert and Thess, 1999)
Side-band Instabilities

Induced Nucleation

\[ v = \varepsilon \sum_{j=1}^{3} A_j e^{i\tilde{q}_j \cdot r} \]
- **Temporal evolution of mean wavevector**

- **transient induced nucleation**
  - ‘persistent’ chaos
  - (Colinet, Nepomnyashchy, and Legros, 2002)

- **Dependence on chiral, nonlinear gradient term** $\alpha_3$

**Separation of dislocations in PHD**

**Limit of persistent creation of PHDs**

- CHIRAL, NONLINEAR GRADIENT TERM $\alpha_3$
Swift-Hohenberg Model

• Rotation and mean flow

\[
\frac{\partial}{\partial t} \psi = R \psi - (\nabla^2 + 1)^2 \psi - \psi^3 + \alpha (\nabla \psi)^2 + \gamma \hat{e}_z \cdot \{ \nabla \psi \times \nabla \Delta \psi \} - \mathbf{U} \cdot \nabla \psi
\]

\[
\nabla^2 \xi = \hat{e}_z \cdot \{ \nabla \psi \times \nabla \Delta \psi \} + \delta \{ (\Delta \psi)^2 + \nabla \psi \cdot \nabla \Delta \psi \} \quad \mathbf{U} = \beta (\partial_y \xi, -\partial_x \xi)
\]

\[R = 0.17, \quad L_x = 233, \quad \alpha = 0.4, \quad \beta = -2.6, \quad \gamma = 2, \quad \delta = 0.0067\]
Defect Proliferation
Defect Statistics in Stripe-Based Defect Chaos

- Complex Ginzburg-Landau equation  
  (Gil, Lega, and Meunier, 1990)
- Electroconvection  
  (Rehberg et al., 1989)
- Convection in inclined layer  
  (Daniels and Bodenschatz, 2002)

- Statistical model for defect pairs  
  (Gil et al., 1990)
  - random pairwise creation
  - uncorrelated diffusion
  - pairwise annihilation upon collision

Squared Poisson distribution for number of defects

$$P(n; \rho) \propto \frac{\rho^n}{(n!)^2}$$
Penta-Hepta Defect Statistics

\[ P(n_{12}^+, n_{12}^-, n_{23}^+, n_{23}^-, n_{31}^+, n_{31}^-) \rightarrow P(n_{12}^+) \equiv P(n) \]

\[ P(n+1) \Gamma^- (n+1) = P(n) \Gamma^+(n) \]

\[ \Gamma^-(n) = a_1 n + a_2 n^2 \]
\[ \Gamma^+(n) = c_0 + c_1 n + c_2 n^2 \]

For \( L = 244 \)

\[ a_1 = \alpha = 8.6 \]
\[ a_2 = 2\beta + \delta \equiv 1 \]
\[ c_0 = \gamma = 20.07 \]
\[ c_1 = 2\alpha = 20.7 \]
\[ c_2 = \beta = 0.12 \]
Creation and Annihilation Rates

- follow defect trajectories: creation and annihilation rates
- consistent with distribution function
- spontaneous creation: locally unstable at low $q$

Defect distribution function

Wavenumber distribution function
Summary:

**Penta-Hepta Defect Chaos in Hexagons**

- Induced Nucleation of Defects
- Proliferation of Defects
- Broad Defect Distribution Function
  - not squared Poisson distribution
- Creation and Annihilation Rates
  - consistent with distribution function
Parametrically Excited Standing Waves

- Coupled Ginzburg-Landau equations
  \[
  \partial_t A + s \partial_x A = d_x \partial_x^2 A + d_y \partial_y^2 A + aA + bB + c|A|^2 A + g|B|^2 A \\
  \partial_t B - s \partial_x B = d_x^* \partial_x^2 B + d_y^* \partial_y^2 B + a^* B + bA + c^*|B|^2 B + g^*|A|^2 B
  \]

Forcing \( f(t) \sim b \)

- Typical phase diagram: periodically forced Hopf bifurcation

![Typical phase diagram](image-url)
- Patterns

- Correlation Functions

**Ordered Chaos**

\[ L = 272 \]
\[ b = 0.7 \]

**Disordered Chaos**

\[ L = 272 \]
\[ b = 0.625 \]
• Defect motion
  - climb ⇒ local change in wavelength
  - glide ⇒ local rotation of pattern
Statistics of Space-Time Loops

- ordered: exponential decay
- disordered: power-law decay
Loop Statistics in a Lattice Model

- Random creation of defect pairs
- Defects diffuse independently on lattice
Loop Statistics in a Lattice Model

- Random creation of defect pairs
- Defects diffuse independently on lattice

\[ \alpha \quad \beta \quad \gamma \]

Coupled CGL \quad 1.5 \quad 3 \quad 2.7

Lattice \quad 1.6 \quad 2.9 \quad 2.4
Single Complex Ginzburg-Landau Equation

- Generic description of Hopf bifurcation
  \[ \partial_t A = A + (1 + i b_1) \nabla^2 A - (b_3 - i) A |A|^2 \]

- Defect chaos above line \( T \): creation and annihilation of defects
  
  (Chaté & Manneville, 1996)

- Defect loop statistics

```latex
\begin{align*}
\text{Number } n \text{ of Defects in Loop} &
\text{Cumul. Relative Frequency} \sim n^{-1.5} \\
\text{x-span } \Delta x &
\text{Cumul. Relative Frequency} \sim \Delta x^{-3} \\
\text{t-span } \Delta t &
\text{Cumul. Relative Frequency} \sim \Delta t^{-2.5}
\end{align*}
```
Defects and Spatio-Temporal Chaos

- Penta-Hepta Defect Chaos
  induced nucleation, proliferation
  broad distribution

  (Y.-N. Young & HR, Physica D, to appear)

- Ordered ⇔ Disordered Spatio-Temporal Chaos
  defect loops in space-time
  exponential vs. algebraic
  defect unbinding transition


- Power Laws in Disordered Regime: Exponents
  - Parametrically driven waves
  - Lattice model
  - Complex Ginzburg-Landau equation

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Parameters

Ginzburg-Landau:
\[ \nu = 2, \alpha_1 = \alpha_2 = 0, \alpha_3 = 0.7, \gamma = 0.2, \tau = 0.5. \]
\[ \beta = 0 \text{ and } \beta = -0.1 \text{ as indicated} \]
large SH:
\[ \alpha = 0.4, \gamma = 2, \beta = -2.6, \delta = 0, R = 0.17, \text{ and } L = 233. \]
small SH:
\[ L = 114, \alpha = 0.4, \gamma = 3, \beta = -5, \delta = 0, R = 0.09. \]
Distribution function in SH: \( L = 244 \)
fit with \( a_1 = 8.6, a_2 = 1 \text{ (normalization)}, c_0 = 20.07, c_1 = 20.7, c_2 = 0.12 \)
update (1/15/2003): the creation rates are actually: \( L=114 \ c_0=31.8 \ c_1=3.3 \ c_2=.4 \ a_1=5 \)
Coupled Ginzburg-Landau equations:
\[ a = +0.25, c = -1 + 4i, d = 1 + 0.5i, s = 0.2, g = -1 - 12i \]
b as indicated
Lattice Model:
\[ L = 1600, p = 0.000016 \]
Farben fuer TEX (aus /usr/local/lib/tex/macros/dvips/colordvi.tex)


\[ a \alpha b \beta a \alpha b \beta a \alpha b \beta \]

\[ a \alpha \]

acroread erst maximieren bevor auf full-screen gegangen wird

check ps2pdf: what’s different

better picture of undulation chaos

warum bekommt \( \alpha_2 \) Term keinen Rotationsterm? warum faktor 2
in Gamma+-
Notes:

KI Faraday mean flow not to be expected to be known
mention only when actually used

PSW: don’t discuss Faraday as the relevant aspect: this is a model that shows a certain transition and it happens to be relevant to Faraday for negative growth rate
spatially extended dynamical systems \rightarrow \text{structured patterened states}

Greenside: RB, excitation waves in heart
disordered in space and chaotic in time

exhibit defects: striking features of pattern

use for characterization and description of spatio-temporal chaos

present results of 2 lines of research

hexagons YY

defect trajectories